Thermoelectric Transport with an 8-Moment Plasma Model in PERSEUS

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April 24th 2018

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NNSA Grant No:
DE-NA0001836

Moments of Boltzmann's Equation

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \partial_{\mathbf{x}} f_\alpha + \frac{\mathbf{F}_\alpha}{m_\alpha} \cdot \partial_{\mathbf{v}} f_\alpha = \frac{\delta f_\alpha}{\delta t} = \sum_\beta C_{\alpha\beta}$$

- Close the moment equations by assuming a form of the distribution *f*
- This limits the degrees of freedom that the distribution could take
- Usually a Maxwellian is assumed (this maximizes entropy, so it's in equilibrium)
- We define physical quantities with the moments that must be consistent with the chosen functional form of *f*

$$f_\alpha(n, \mathbf{v}, T) = f_0 = n \left(\frac{m}{2\pi k_B T} \right)^{\frac{3}{2}} \exp\left(\frac{-mv^2}{2k_B T}\right)$$

$$\mathbf{w}_\alpha = \mathbf{v}_\alpha - \mathbf{u}_\alpha \qquad n_\alpha = < f_\alpha > \qquad T_\alpha = \frac{m_\alpha < w_\alpha^2 f_\alpha >}{3 < f_\alpha >}$$

$$< \mathbf{w}_\alpha f_\alpha > = 0 \qquad \mathbf{u}_\alpha = \frac{< \mathbf{v}_\alpha f_\alpha >}{< f_\alpha >}$$

5-Moment model

$$f_\alpha(n, \mathbf{v}, T) = f_0 = n\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} \exp\left(\frac{-mv^2}{2k_B T}\right)$$

$$\partial_t \rho + \nabla \cdot \rho\mathbf{u} = 0 \qquad \rho = \sum m_\alpha n_\alpha$$

$$\partial_t(\rho\mathbf{u}) + \nabla \cdot (\rho\mathbf{u}\mathbf{u} + \mathbf{P}) = \mathbf{J} \times \mathbf{B}$$

$$\partial_t \epsilon_\alpha + \nabla \cdot \mathbf{Q}_\alpha = Z_\alpha n_\alpha \mathbf{u}_\alpha \cdot \mathbf{E} \qquad \epsilon_\alpha = \frac{1}{2} m_\alpha < v^2 f_\alpha >$$

$$\mathbf{Q} \equiv (\epsilon + P)\mathbf{u} + \mathbf{q} \qquad \boxed{\mathbf{q}_\alpha = \frac{1}{2} m_\alpha < \mathbf{w}_\alpha w_\alpha^2 f_\alpha > = 0}$$

- Assumes equilibrium: no heat flux or stress
- Corresponds to a Gaussian distribution

The resulting Generalized Ohm's Law (GOL):

$$\partial_t \mathbf{J} + \nabla \cdot \left(\mathbf{uJ} + \mathbf{Ju} - \frac{\mathbf{JJ}}{ne} - \frac{e}{m_e}\mathbf{P}_e\right) = \frac{ne^2}{m_e}\left(\mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{\mathbf{J} \times \mathbf{B}}{ne}\right) - \nu_{ei}\mathbf{J}$$

Pressure is isotropic if a Maxwellian distribution is assumed

There is no 'thermal force' term

- If we want to describe a more general plasma we must have a more general *f :*

$$f_\alpha = \left(1 - \frac{m^2 w^2}{5 k_B^2 T^2 P}\left(1 - \frac{m w^2}{7 k_B T}\right)\mathbf{q}\cdot\mathbf{w}\right) f_0$$ *

- Can now be non-equilibrium; includes heat flux
- Corresponds to a Gaussian with a skew
- Includes the physics of thermal conductivity and Nernst

$$\partial_t \mathbf{J} + \nabla\cdot\left(\mathbf{uJ} + \mathbf{Ju} - \frac{\mathbf{JJ}}{ne} - \frac{e}{m_e}\mathbf{P}_e\right) = \frac{ne^2}{m_e}\left(\mathbf{E} + \mathbf{u}\times\mathbf{B} - \frac{\mathbf{J}\times\mathbf{B}}{ne}\right) - \nu_{ei}\mathbf{J} \boxed{- \frac{6}{35}\frac{e\nu_{ei}}{T_e}\mathbf{q_e}}$$

$$\partial_t \mathbf{Q}_\alpha + \nabla\cdot\mathbf{G}_\alpha = \frac{Z_\alpha}{m_\alpha}\left((\epsilon_\alpha + P_\alpha)\mathbf{E} + \mathbf{Q}_\alpha\times\mathbf{B}\right) - \nu_2\mathbf{q}_\alpha - \frac{T_\alpha \nu_{\alpha\beta}}{Z_\alpha}\mathbf{J}$$

$$\mathbf{G} \equiv (\epsilon + 2P)\mathbf{uu} + \frac{7}{5}(\mathbf{uq} + \mathbf{qu}) + \left(\frac{1}{2}u^2 P + \frac{2}{5}\mathbf{u}\cdot\mathbf{q} + \boxed{\frac{5P^2}{2\rho}}\right)\mathbf{I}$$

* M. Killie et al., ”Improved Transport Equations For Fully Ionized Gases”, The Astrophysical Journal, 604:842-849, 2004

- 13 moments would give heat flux *and* stress

$$f_\alpha = f_{0,\alpha}[1 + \frac{m_\alpha}{2T_\alpha P_\alpha}\mathbf{S}_\alpha : \mathbf{w}_\alpha\mathbf{w}_\alpha - \frac{m_\alpha^2 w_\alpha^2}{5T_\alpha^2 P_\alpha}(1 - \frac{m_\alpha w_\alpha^2}{7T_\alpha})\mathbf{q}_\alpha \cdot \mathbf{w}_\alpha]$$

$$\mathbf{S}_\alpha = \mathbf{P}_\alpha - P_\alpha\mathbf{I}$$

- Corresponds to a Gaussian with different widths in different directions (non-isotropic) as well as a skew
- This functional form looks complicated, but gives consistent definitions for all 13 variables
- Would include additional physics such as shear and viscosity

- Going beyond 13-moments? More complicated functional form
- Small corrections to non-equilibrium physics
- Law of diminishing returns
- Equations become very complicated and harder to implement

Alternative to the moment approach

- Braginskii (1965) used the perturbative technique of Chapman & Cowling:

$$f = f_0 + \epsilon f_1$$

- **Assumes near-equilibrium;** variables change slowly in time and space

$$\frac{d}{dt} \ll \frac{1}{\tau} \qquad l \ll L \qquad \epsilon \ll 1$$

- Solve for f_1 by dropping terms chosen to be small, rather than choosing f
- Results in the equations found in the NRL formulary

* Braginskii, S. I., ”Transport Processes in a Plasma”, Reviews of Plasma Physics, 1, 205-311, 1965

- These resulting source terms involve gradients:

Thermal Force: $$\mathbf{R}_T = -0.71 n \nabla_{||}(k_B T) - \frac{3n}{2\omega\tau}\mathbf{b} \times \nabla_{\perp}(k_B T)$$

Thermal Conductivity: $$\mathbf{q}_T = -\kappa \nabla (k_B T)$$

- Because the time derivative terms were dropped, Braginskii has infinite wave speeds for non-equilibrium variables, like heat propagation, which in unphysical

- They are also harder to solve because they are not in "conservation form":

$$\frac{\partial \mathbf{X}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{X}) = \mathbf{S}(\mathbf{X})$$

Solving the equations with Perseus

- We solve the conservative 8-moment equations (non-equilibrium)
- This involves a **local** implicit solve for Ohm's Law
- The Heat flow equation looks a lot like Ohm's Law, and can be solved with a similar local implicit solve
- Perseus is very fast, and can handle large source terms like Hall (at low n) and Nernst (at high n)

Ampere's law

$$\mathbf{E}^{n+1} = \mathbf{E}^* + \frac{\Delta t}{\delta}\left(\nabla \times \mathbf{B}^* - \mu_0 \mathbf{J}^{n+1}\right)$$

Ohm's Law

$$\mathbf{J}^{n+1} = \mathbf{J}^* + \frac{\Delta t}{\epsilon}\left(\mathbf{E}^{n+1} + \mathbf{u}^* \times \mathbf{B}^* - \underbrace{\frac{1}{n_e^* e}\mathbf{J}^{n+1} \times \mathbf{B}^*}_{\text{Hall effect}} - \underbrace{\eta^* \mathbf{J}^{n+1}}_{\text{Resistivity}}\right) \boxed{-\Delta t \frac{6e\nu_{ei}}{35 T_e^*}\mathbf{q}^{n+1}}$$

Thermal force

$$\mathbf{q}^{n+1} = \mathbf{q}^* - \frac{\Delta t}{\delta}\left(\mathbf{E}^{n+1} + \mathbf{u}^* \times \mathbf{B}^* - \frac{1}{n_e^*}\mathbf{J}^{n+1} \times \mathbf{B}^*\right) - \Delta t \omega_{c,e} \mathbf{q}^{n+1} \times \mathbf{B}^* - \Delta t \nu_2 \mathbf{q}^{n+1} + \Delta t \alpha_2 \mathbf{J}^*$$

Heat flow equation

The moment formulation can be shown to reduce to Braginskii if the same near-equilibrium assumption is made

For example, the Nernst effect comes from the moments:

Make the same approximations as Braginksii: $\Omega_e >> \partial_t$

Simplify the equations by letting $\hat{\mathbf{b}} = \hat{\mathbf{z}}$ and look at the $\mathbf{q_e}\hat{\mathbf{y}}$ equation

The heat flow moment equation reduces to:

$$\mathcal{O}\left(\frac{\partial_t}{\Omega_e}\right) \approx 0 \approx \frac{5P_e}{2m_e\Omega_e}\partial_y T_e - q_e \times \widehat{b} + \frac{3\nu_{ei} T_e}{2e\Omega_e} j_y$$

$$q_e \approx \frac{5P_e}{2m_e\Omega_e}\partial_y T_e + \frac{3\nu_{ei} T_e}{2e\Omega_e} j_y$$

We saw that the 8-moment model gave rise to a “thermal force” for the electrons:

$$\mathbf{R}_T = -\frac{6e\nu_{ei}}{35T_e^*}\mathbf{q}^{n+1}$$

This results in a force perpendicular to both the thermal gradient and the field (ie. the Nernst effect)

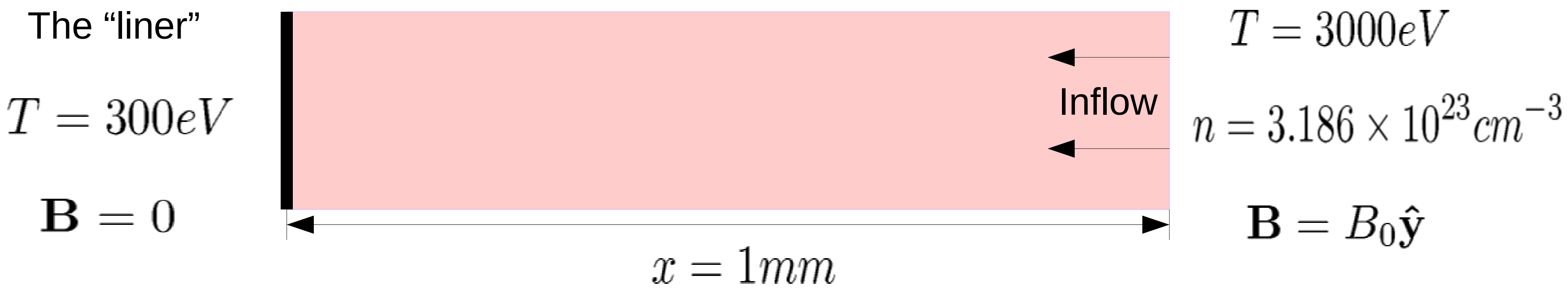
$$\boxed{\mathbf{R}_{T,\,\wedge} = -\frac{3\nu_{ei} n}{7\Omega_e}(\widehat{b} \times \nabla T_e)}$$

Verifying the 8-moment model

- Velikovich, Giuliani & Zalesak (2015) came up with an analytical test for codes that include the Nernst effect
- 2 ns problem of MagLIF at plasma stagnation on the liner
- They found that the Nernst term has a dramatic effect on the energy losses
- They made a custom 1D Lagrangian code that matched their results extremely well
- Other codes that incorporate Nernst can compare their results to this test

* Velikovich, A. L., Giuliani, J. L., Zalesak, S. T., ”Magnetic flux and heat losses by diffusive, advective, and Nernst effects in magnetized liner inertial fusion-like plasma”, Physics of Plasmas, 22, 042702, 2015

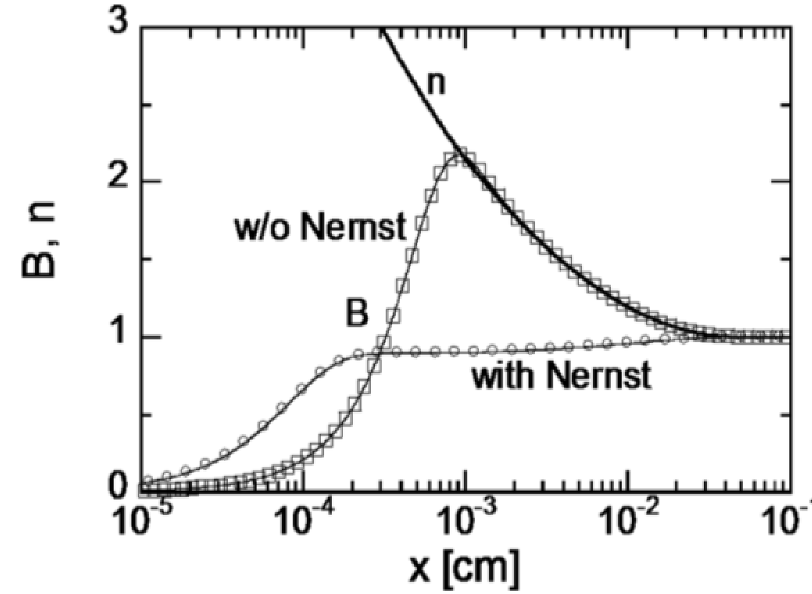
1-dimensional problem:

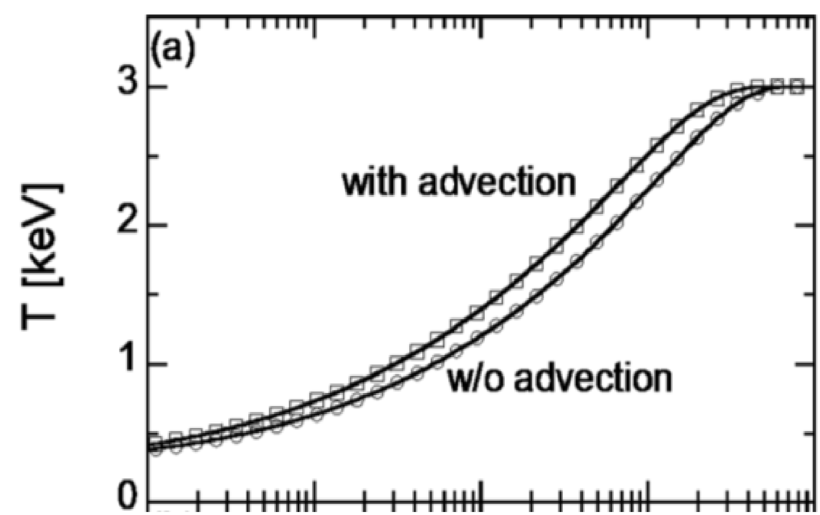


- The system is a hot, dense, magnetized deuterium plasma stagnated against a cold, unmagnetized liner at the left boundary
- As thermal & magnetic energy is deposited into the liner, the pressure in the system decreases, allowing inflow of plasma from the right boundary
- The analytical test found a self-similar solution to this problem (reducing the PDEs to ODEs) in 1D
- They assumed an infinite speed of sound for simplicity. No ionization (deuterium) or radiation effects

Numerical results

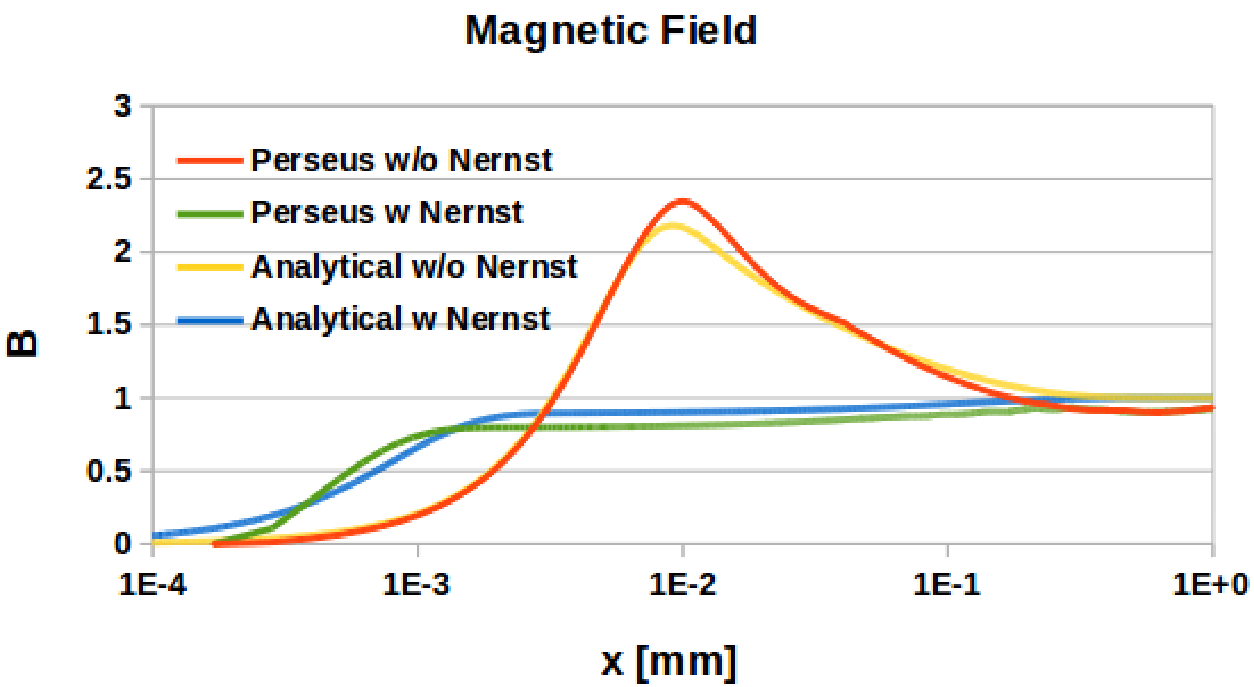
Self-similar analytical solutions

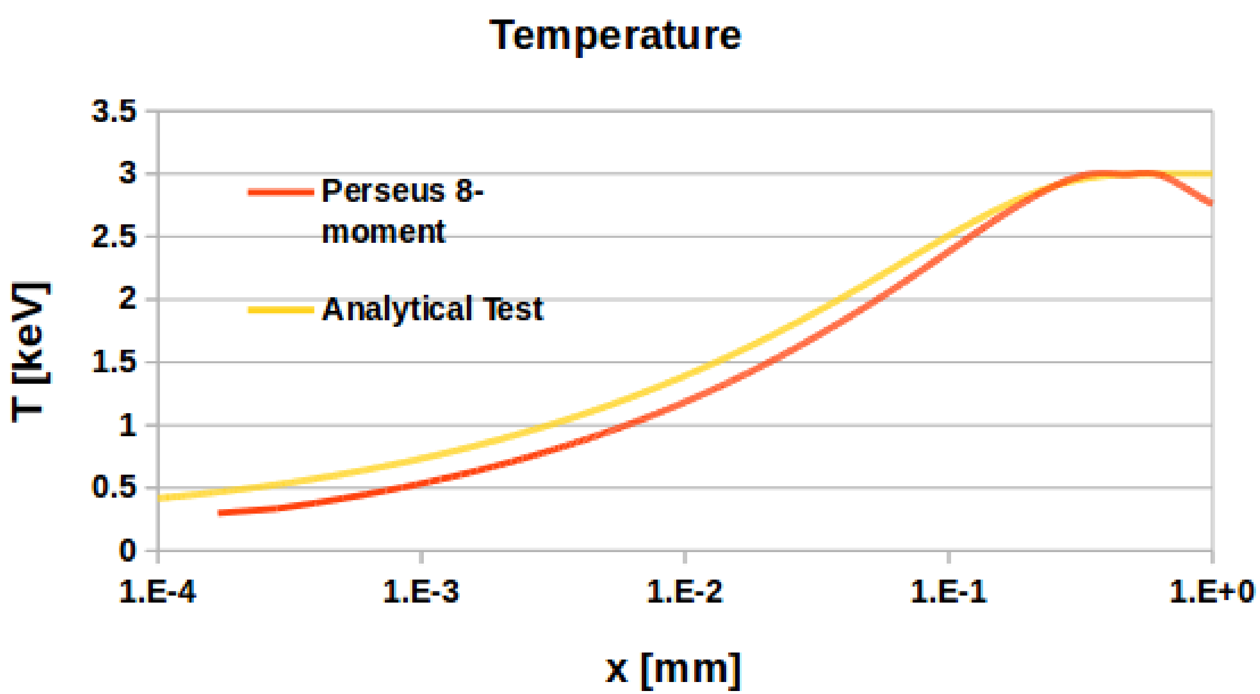




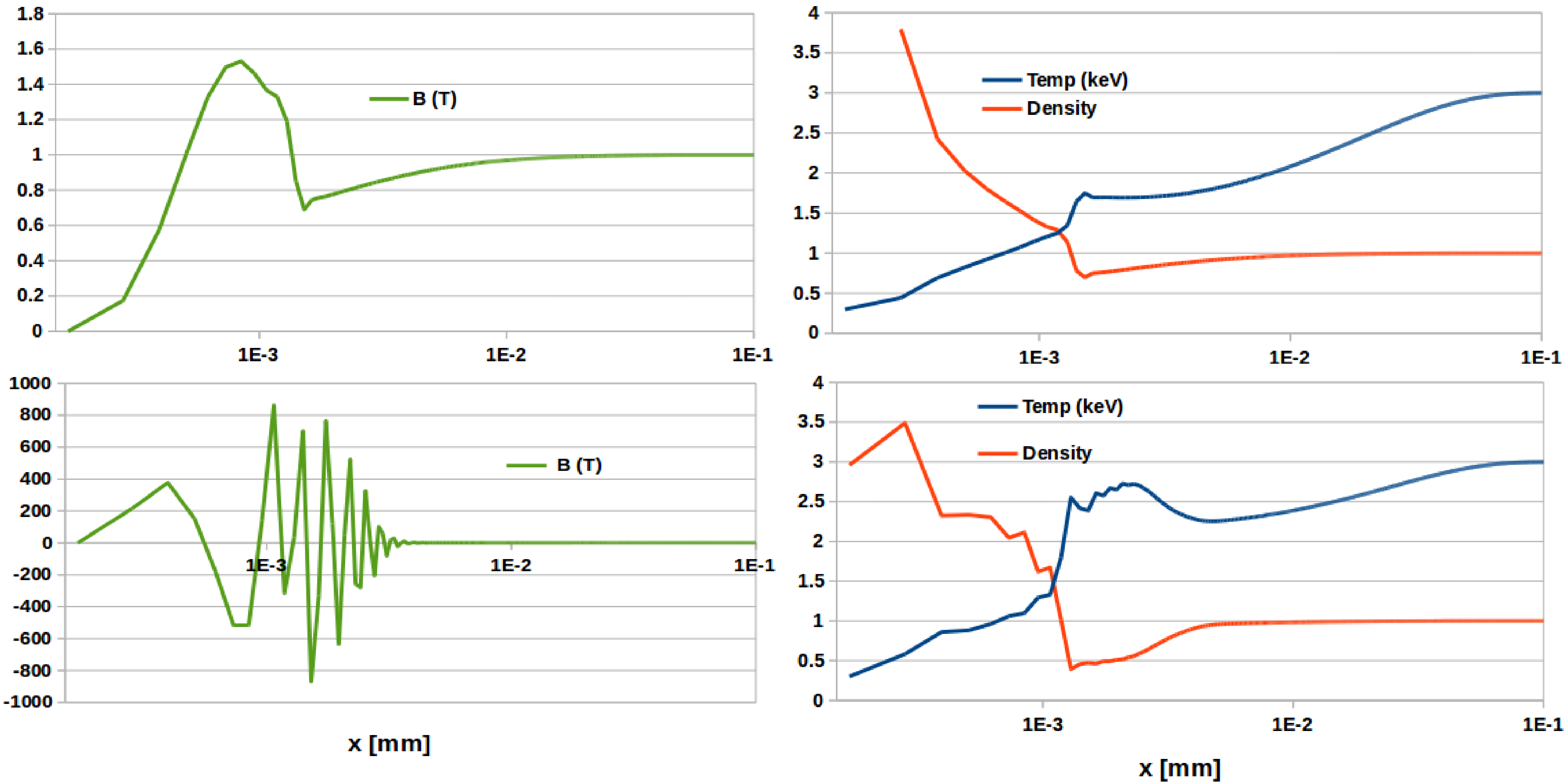
- As density piles up on the liner, B is pulled in as well (frozen-in flux theorem), but is lost to the wall due to resistivity
- In this geometry, Nernst acts like a resistive term in Faraday's Law, increasing the dissipation into the liner and further decoupling B from the density
- Thermal conductivity plays an analogous role to resistivity for the temperature
- As far as we know, only LASNEX has “passed” their test, but this code is “classified”
- We don't know in which ways other codes have failed, or in which way(s) LASNEX has passed

* Velikovich, A. L., Giuliani, J. L., Zalesak, S. T., ”Magnetic flux and heat losses by diffusive, advective, and Nernst effects in magnetized liner inertial fusion-like plasma”, Physics of Plasmas, 22, 042702, 2015


Magnetic Field
Perseus w/o Nernst
Perseus w Nernst
Analytical w/o Nernst
Analytical w Nernst
B
0
0.5
1
1.5
2
2.5
3
1E-4
1E-3
1E-2
1E-1
1E+0
x [mm]


Temperature
Perseus 8-moment
Analytical Test
T [keV]
0
0.5
1
1.5
2
2.5
3
3.5
1.E-4
1.E-3
1.E-2
1.E-1
1.E+0
x [mm]

Numerical Stability of this method remains a problem:



$$\partial_t \mathbf{Q}_\alpha + \nabla \cdot \mathbf{G}_\alpha = \frac{Z_\alpha}{m_\alpha}((\epsilon_\alpha + P_\alpha)\mathbf{E} + \mathbf{Q}_\alpha \times \mathbf{B}) \boxed{- \nu_2 \mathbf{q}_\alpha} - \frac{T_\alpha \nu_{\alpha\beta}}{Z_\alpha}\mathbf{J}$$

$$\mathbf{G} \equiv (\epsilon + 2P)\mathbf{uu} + \frac{7}{5}(\mathbf{uq} + \mathbf{qu}) + (\frac{1}{2}u^2 P + \frac{2}{5}\mathbf{u}\cdot\mathbf{q} \boxed{+ \frac{5P^2}{2\rho})\mathbf{I}}$$

Conclusions

- Moment formulation has the advantages of conservation form (easier solves), non-equilibrium flexibility, physical propagation speeds

- Has been shown to agree with the analytical results

- Moving forward, we are studying the numerical stability of the equations in Perseus, and looking for ways to make them more robust

Conservation form

$$\frac{\partial \mathbf{X}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{X}) = \mathbf{S}(\mathbf{X})$$

- No gradients in the source terms
- Ensures the variables are conserved to high accuracy
- Notice that all moment equations are conservative, because Boltzmann's equation is
- Braginskii's equations are **not** in conservation form!

If a source term is large it must be updated implicitly:

$$\frac{\partial \mathbf{U}}{\partial t} = -\nu \mathbf{U}$$

Explicitly

$$\mathbf{U}^{n+1} = \mathbf{U}^{n} - dt\, \nu \mathbf{U}^{n}$$

$$\mathbf{U}^{n+1} = \mathbf{U}^{n}(1 - dt\, \nu)$$

Implicitly

$$\mathbf{U}^{n+1} = \mathbf{U}^{n} - dt\, \nu \mathbf{U}^{n+1}$$

$$\mathbf{U}^{n+1} = \mathbf{U}^{n}/(1 + dt\, \nu)$$

After m iterations, errors grow as:

$$\left|\frac{\delta \mathbf{U}^{n+m}}{\delta \mathbf{U}^{n}}\right| = |(1 - dt\, \nu)|^{m} \rightarrow \infty$$

$$\left|\frac{\delta \mathbf{U}^{n+m}}{\delta \mathbf{U}^{n}}\right| = |1/(1 + dt\, \nu)|^{m} \rightarrow 0$$

Unless $dt\, \nu < 1$

NOT SATISFIED FOR LARGE SOURCE TERMS!

Regardless of $dt\, \nu$

- Gradients in the source terms make an equation non-local and non-conservative

$$S(\nabla^2 X_i) = S(\frac{X_{i+1} - 2X_i + X_{i-1}}{dx^2})$$

- The implicit solve for those source terms becomes very hard, and could take ~90% of a simulation's run time

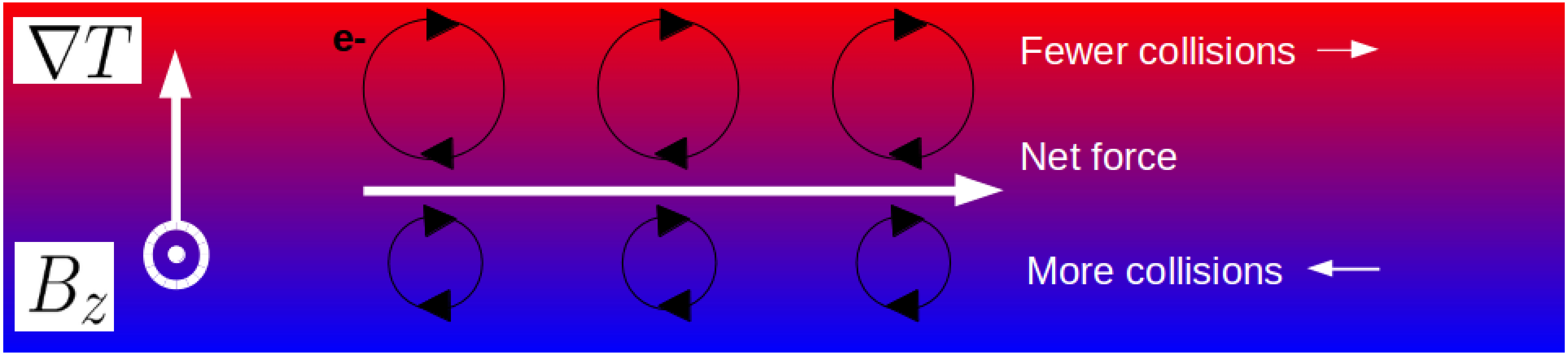
- Completely local source terms are relatively much easier and quicker to update (hence why we like conservation form)

- Braginskii's equations are useful to reference, but not convenient when trying to solve a partial differential equation with high accuracy methods (ie. Finite Volume or Discontinuous Galerkin)

Experimental Validation?

- As far as I'm aware, no one has done an experiment to validate Velikovich & Giuliani's test
- Nernst is dominate at high densities and high temperature gradients
- The analytical test tells us that Nernst greatly increases the loss of energy from the plasma into the liner, but can be partially mitigated by magnetizing the plasma
- There is a lot of physics not included in either of our approaches that would be present in an experiment

What is the Nernst Effect?



In a plasma: $\nu_{ei} \sim T_e^{-\frac{3}{2}}$

The resulting force on the electrons is: $$\mathbf{R}_{T,\perp} = -\frac{3\nu_{ei} n}{7\Omega_e}(\hat{b} \times \nabla T_e)$$

Thought to be important for magnetic+thermal energy dissipation in MagLIF