Toroidal Equilibria with Transonic Poloidal Flow

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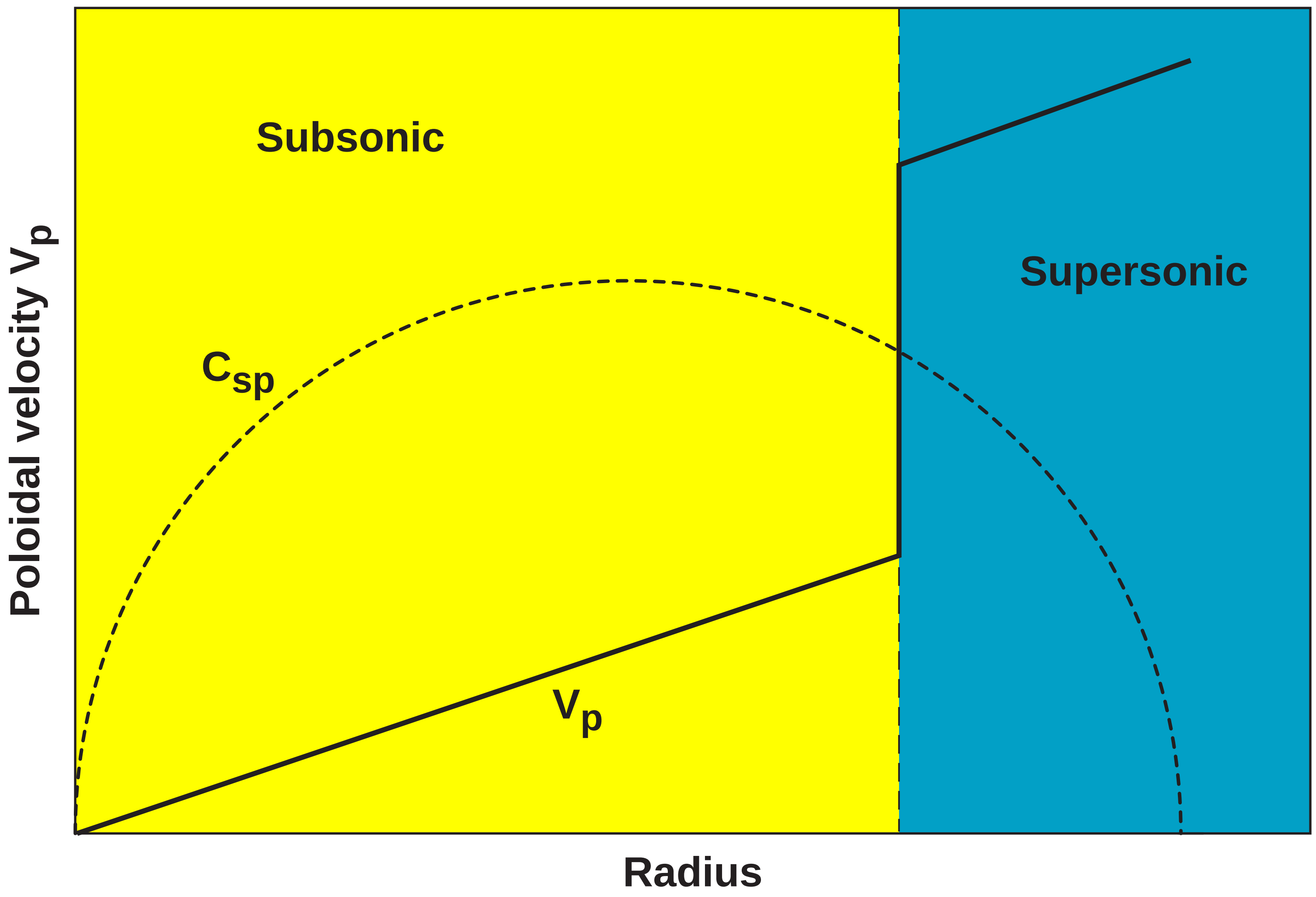
Summary

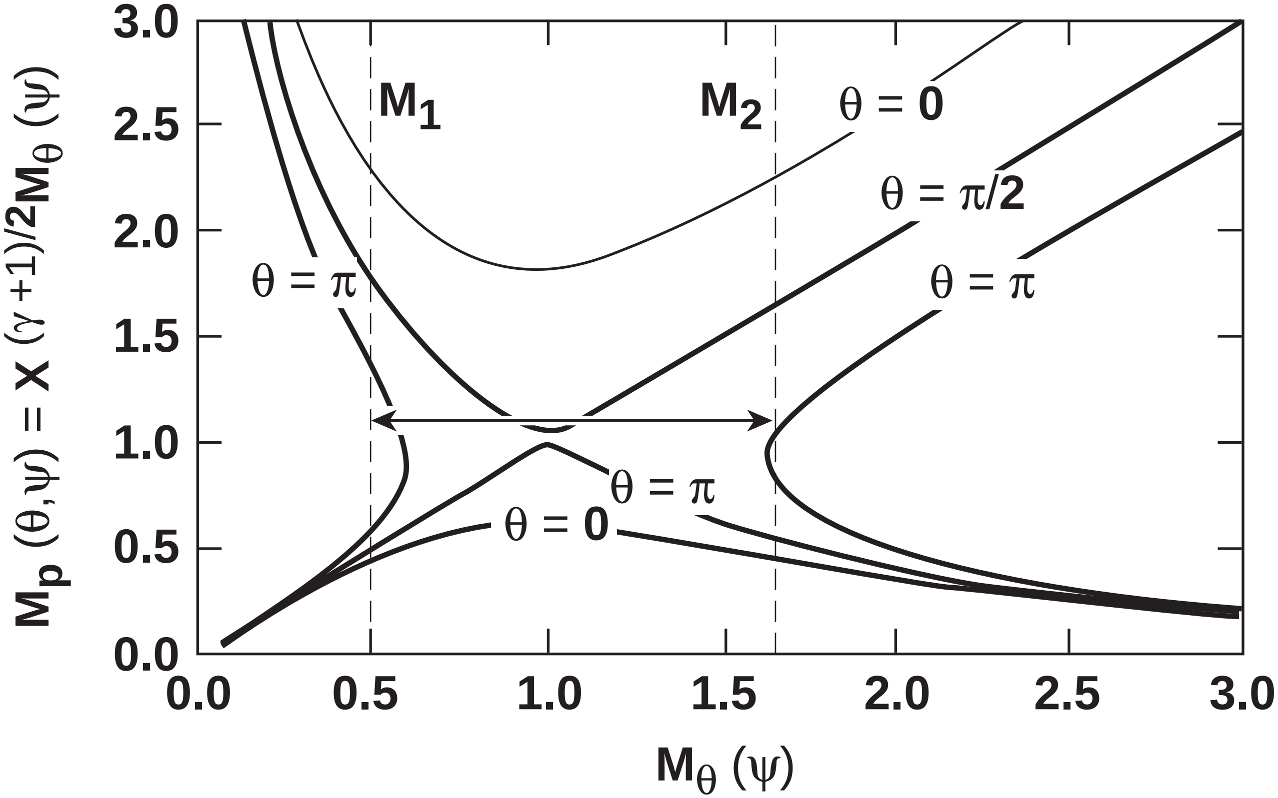
Transonic equilibria are radially discontinuous



TC5156

The solution of the Bernoulli equation yields a radially discontinuous poloidal Mach number M_p at θ ? π

- Poloidal Mach number M_p versus free function $M_\theta(\psi)$.
- No solutions at $\theta = \pi$ for $M_1 < M_\theta < M_2$.
- The free function M_θ must be $< M_1$ (or $> M_2$).
- The poloidal Mach number must be radially discontinuous.



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MHD equilibrium equations with arbitrary flow

$$\nabla \bullet (\rho \vec{\mathbf{v}}) = \mathbf{0}$$

$$\rho \vec{\mathbf{v}} \bullet \nabla \vec{\mathbf{v}} = -\nabla \mathbf{P} + \vec{\mathbf{J}} \times \vec{\mathbf{B}} + \nabla \nu_\theta \nabla \vec{\mathbf{v}}$$

$$\nabla \bullet (\rho \mathbf{S} \vec{\mathbf{v}}) = \nu_\theta (\mathbf{e}_\theta \bullet \nabla \mathbf{v}_\theta)^{\mathbf{2}} / \mathbf{T}$$

$$\mathbf{p} = (\mathbf{1} + \mathbf{Z}) \rho \mathbf{T} / \mathbf{m_i}$$

$$\mathbf{S} = \mathbf{c_v} \, \mathbf{log}\left(\mathbf{p}/\rho^\gamma\right)$$

$$\mu_{\mathbf{0}} \vec{\mathbf{J}} = \nabla \times \vec{\mathbf{B}}$$

$$\nabla \bullet \vec{\mathbf{B}} = \mathbf{0}$$

$$\nabla \times \left(\vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) = \mathbf{0}$$

Look for continuous solutions in $\theta \rightarrow$ neglect viscosity; the MHD equilibrium depends on five free functions

$$\nabla \bullet \vec{B} = 0 \longrightarrow \vec{B}_p = \frac{1}{R} \nabla \Psi \times \vec{e}_\varphi$$

$$\nabla \times \left(\vec{v} \times \vec{B} \right) = 0 \longrightarrow \vec{v} = v_{||} \vec{b} - \Omega(\Psi) R \vec{e}_\varphi$$

$$\nabla \bullet \rho \vec{v} = 0 \longrightarrow \rho = D(\psi) \big/ \left(v_{||} / B \right)$$

$$\nabla \bullet \left(\rho S \vec{v} \right) = 0 \longrightarrow \nabla \bullet p^{1/\gamma} \vec{v} = 0 \longrightarrow p = Q(\psi) \big/ \left(v_{||} / B \right)$$

φ**-momentum** $\rightarrow$ $$B_\varphi R = F(\Psi) + D(\Psi) v_\varphi R$$

V-momentum: $\rightarrow$ $$\rho \left[\frac{v^2}{2} + \Omega(\Psi) R v_\varphi \right] + \frac{\gamma}{\gamma - 1} p = \frac{N(\Psi)}{\left(v_{||} / B \right)}$$

→ **$D(\psi)$, $Q(\psi)$, $F(\psi)$, $\Omega(\psi)$, $N(\Psi)$ are free functions of Ψ determined by the transport equations.**

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$\nabla\psi$–component of momentum = Grad-Shafranov

$$\Delta^{*}\psi + \mathbf{e}_{\psi} \bullet \nabla\left(B_{\phi}^{2}R^{2}/2\right) + R^{2}\mathbf{e}_{\psi} \bullet \nabla p +$$

$$-\left(\rho v_{\phi}^{2}/B_{p}^{2}\right)B_{Z} = O\left(\rho v_{p}^{2}/B_{p}^{2}\right) \sim \in^{2}$$

Ordering:

$\in = a/R_0 < 1$

$\beta \sim \in^{2}$, $B_p/B_\phi \sim \in$,

$v_p \sim \in C_s$,

$v_\phi \sim \left(\in C_s \text{ or } C_s\right)$

$C_{sp} = \dfrac{B_p}{B} C_s =$ poloidal sound speed

$M_p = \dfrac{v_p}{C_{sp}} \sim 1 =$ poloidal Mach number

The Bernoulli equation must be solved to find the poloidal Mach number M_p

Definitions:

$$\rho = D(\psi)/X$$

$$p = P(\psi)/X^{\gamma}$$

$$\Delta^{*}\psi + e_{\psi} \bullet \nabla\left(R^2 p + B_{\phi}^2 R^2/2\right) = O(\in)$$

$$C_s(\psi) = \sqrt{\gamma P(\psi)/D(\psi)}$$

$$v_p = X\, C_s(\psi)\, M_{\theta}(\psi)\, B_p/B$$

$$v_{\phi} \cong C_s(\psi)\left[M_{\phi}(\psi) + (X-1) M_{\vartheta}(\psi)\right]$$

$$B_{\phi} R = F(\psi)\left[1 + O\left(\in^2 X\right)\right]$$

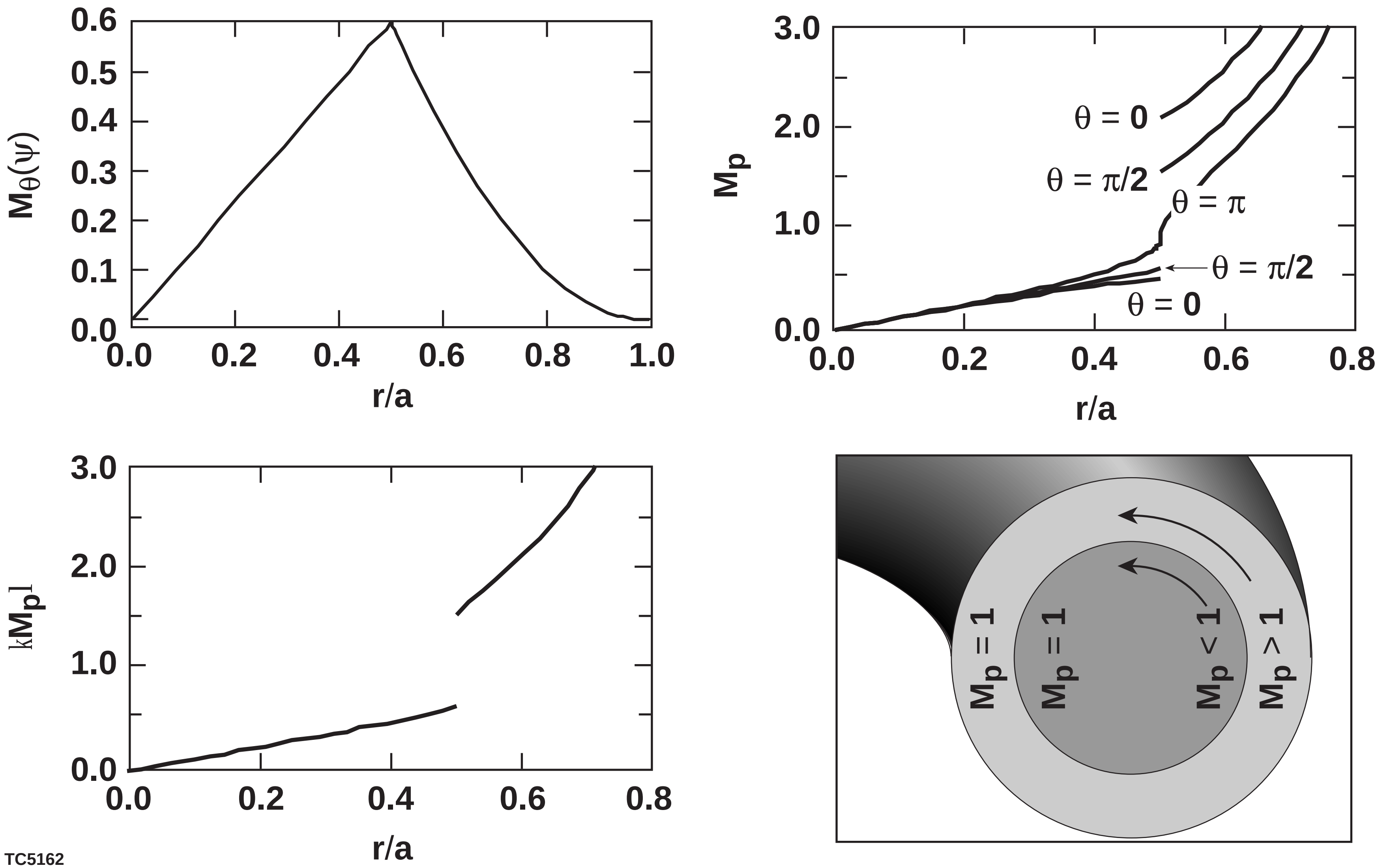
Bernoulli equation

$$\breve{R}^{-2} M_{\vartheta}^2(\psi) X^{\gamma+1} - \left\{\frac{2}{\gamma-1} + M_{\vartheta}^2(\psi) + \left[M_{\vartheta}(\psi) - M_{\phi}(\psi)\right]\left(\breve{R}^2 - 1\right)\right\} X^{\gamma-1} + \frac{2}{\gamma-1} = 0$$

$$\breve{R} = 1 + \in \cos\vartheta \qquad X = X(\psi, \vartheta) \qquad M_p = X^{(\gamma+1)/2} M_{\theta}(\psi)$$

$P(\psi)$, $D(\psi)$, $F(\psi)$, $M_{\phi}(\psi)$, and $M_{\theta}(\psi)$ are new free functions.

Shockless transonic equilibria have discontinuous Mach number profiles



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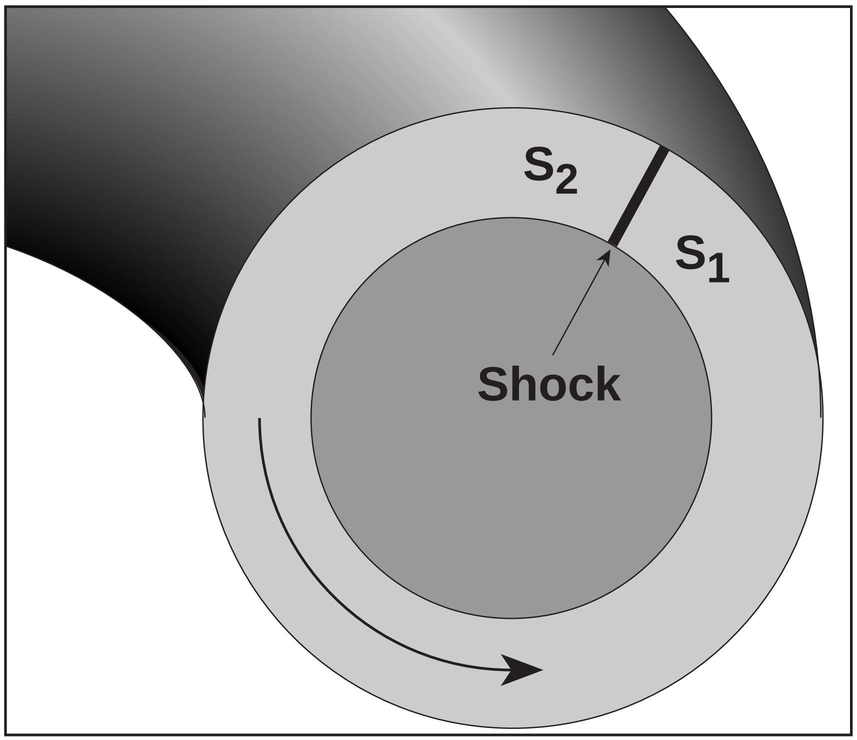
Shocked solutions do not represent “real” equilibria

- Sources of irreversibility cannot be stationary in a periodic system.



S = entropy

$S_2 > S_1$

$= \bullet (\rho S\vec{v}) = \nu_q (e_q \bullet = v_q)^2/T$

$\Delta S \sim \in^{3/2} S$

- If the shock were stationary, the entropy would constantly increase.

- The shock ***must travel:*** $\theta_s = \theta_s(t)$

- The shock velocity is slow: $\dot{\theta}_s(t) \sim \in^{1/2} \frac{v_\theta}{r}$

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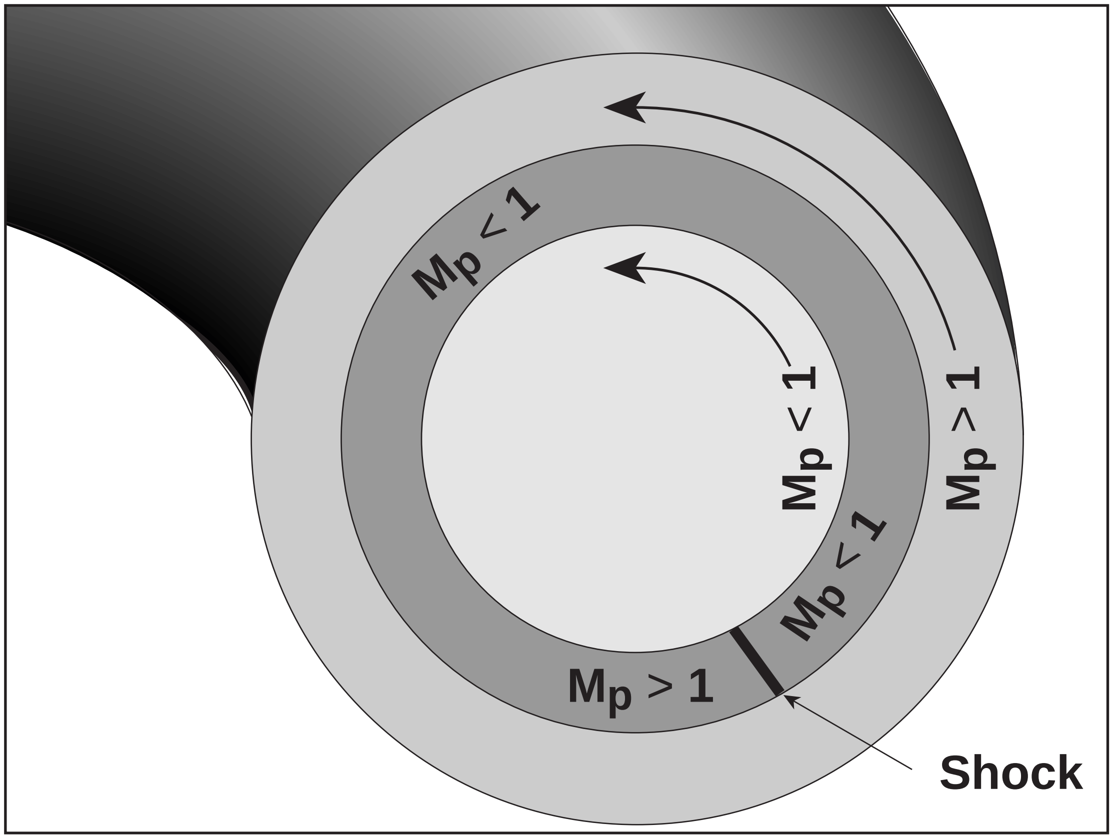
Can equilibria with shocks exist?



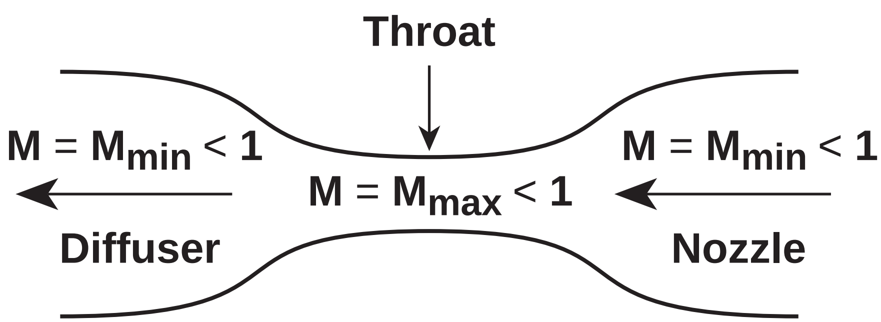
Hazeltine, Lee, Rosenbluth (1971); Shaing, Hazeltine, and Sanuki (1992).

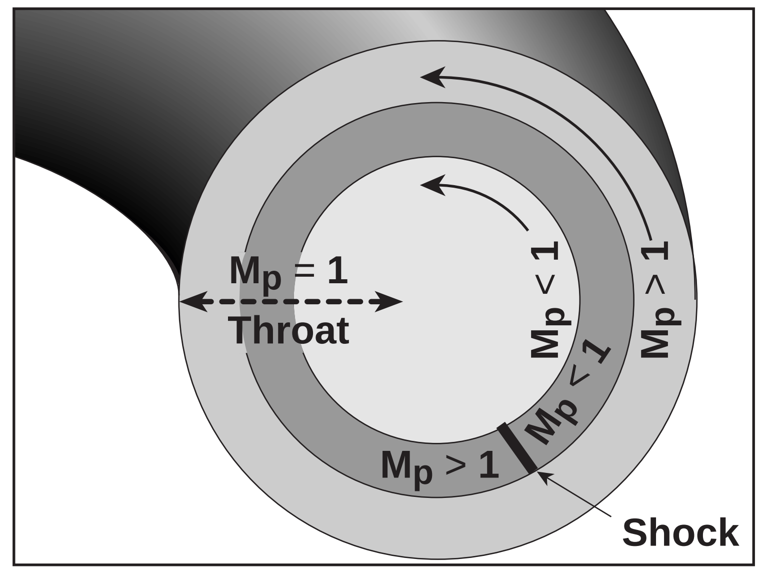
Shock dynamics

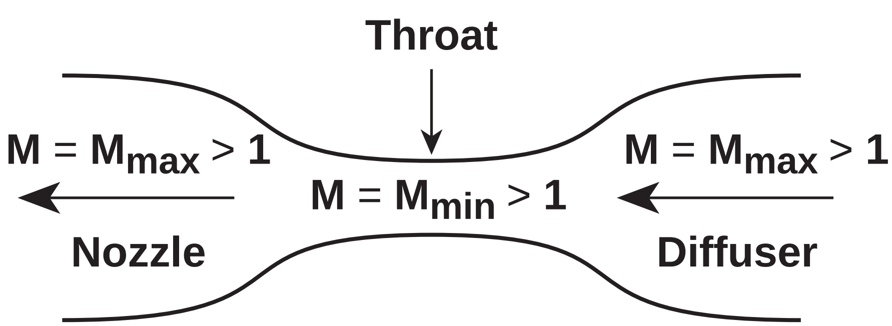
- Shocks cannot persist at a steady state.
- If a shock forms, it weakens while moving poloidally toward the inner midplane.
- The shock vanishes after reaching the inner midplane.
- Traveling shocks leave behind the radially discontinuous equilibria previously described.

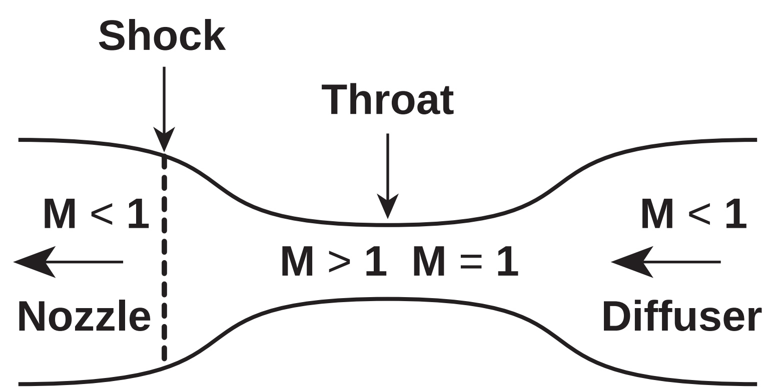
The tokamak magnetic flux surfaces act as a nozzle/diffuser

- The flow is either "well" supersonic or "well" subsonic.









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Summary

- In the presence of transonic poloidal flow, the isentropic MHD equations yield two classes of equilibria:
  - shockless equilibria with radially discontinuous profiles and
  - equilibria with weak shocks.

- Because of the finite irreversibility, the isentropic MHD model does not correctly describe shocks (weak or strong) in a periodic system.

- The energy-conserving (including the viscous heating) MHD model shows that shocked equilibria evolve into shockless radially discontinuous equilibria.

The shock trajectory $\theta_s(t)$ can be determined by the solvability conditions of the time-dependent MHD equations

Example:

$$\nabla \bullet \left(\rho \vec{v}_p\right) = -\partial_t \rho \qquad \partial_t \sim \in^{1/2} \frac{v_p}{r}$$

$$\rho = \rho_0 + \in^{1/2} \rho_{1/2} + \in \rho_1 \ldots\ldots$$

$$\vec{v}_p = \vec{v}_{p0} + \in^{1/2} \vec{v}_{p(1/2)} + \in \vec{v}_{p1} \ldots\ldots$$

$$\begin{cases} \in^0 - \textbf{order} \\ \nabla \bullet \left(\rho_0 \vec{v}_{p0}\right) = 0 \rightarrow \vec{B}_p \bullet \nabla\left(\rho_0 v_{p0}/B_p\right) = 0 \rightarrow \rho_0 v_{p0}/B_p = D_0\left[\psi, \theta_s(t)\right] \end{cases}$$

$$\begin{cases} \in^{1/2} - \textbf{order} \\ \nabla \bullet \left[\rho_{1/2} \vec{v}_{p0} + \rho_0 \vec{v}_{p(1/2)}\right] = -\partial_t \rho_0 \\ \vec{B}_p \bullet \nabla\left[\left(\rho_{1/2} v_{p0} + \rho_0 v_{p(1/2)}\right)\big/ B_p\right] = -\partial_t \rho_0 \end{cases} \longrightarrow \int_{-\pi}^{+\pi} \frac{\partial_t \rho_0}{\vec{B}_p \bullet \nabla\theta} d\theta = 0$$

$$\begin{cases} \in^1 - \textbf{order} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

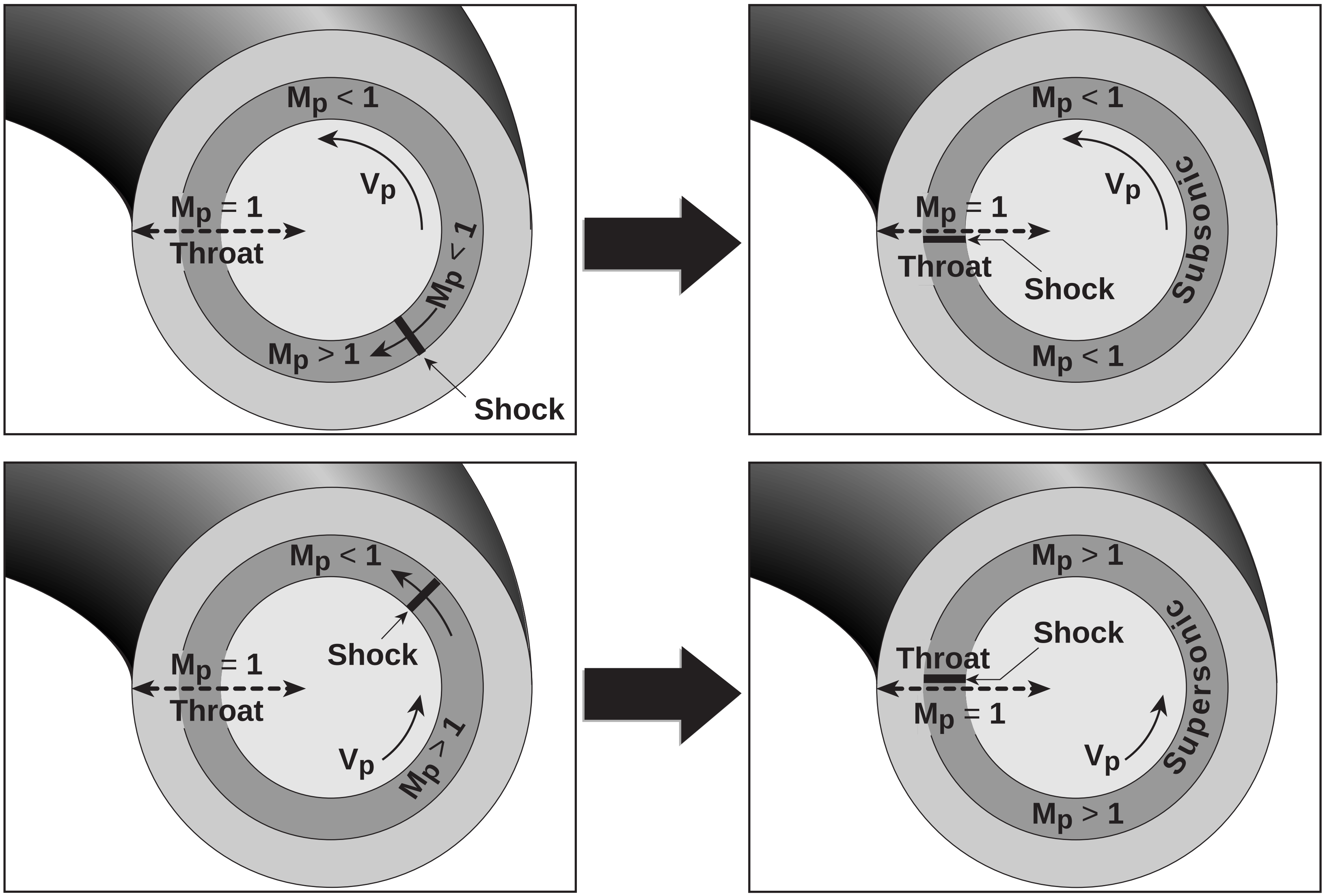
Shock poloidal trajectory

$$\frac{d\theta_s}{dt} = \frac{\pi}{3}\left[2(\gamma+1)\in\right]^{1/2} \frac{C_{sp}}{r} \frac{\cos^3(\theta_s/2)}{\sin\theta_s}$$

- If $\theta_s(0) > 0$, then $\dot{\theta}_s(t) > 0$ and the shock travels counterclockwise.
- If $\theta_s(0) < 0$, then $\dot{\theta}_s(t) < 0$ and the shock travels clockwise.
- The shock stops when it reaches the inward midplane $\theta_s = \pm\pi$.
- The shock disapperars at the inward midplane:

$$\Delta M_p\Big|_{\pm\pi} = 2\sqrt{2(\gamma+1)\in}\,\cos\left(\frac{\theta_s}{2}\right)\Bigg|_{\pm\pi} = 0$$

Shocks in the upper/lower mid-plane travel leaving behind a super/subsonic flow



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