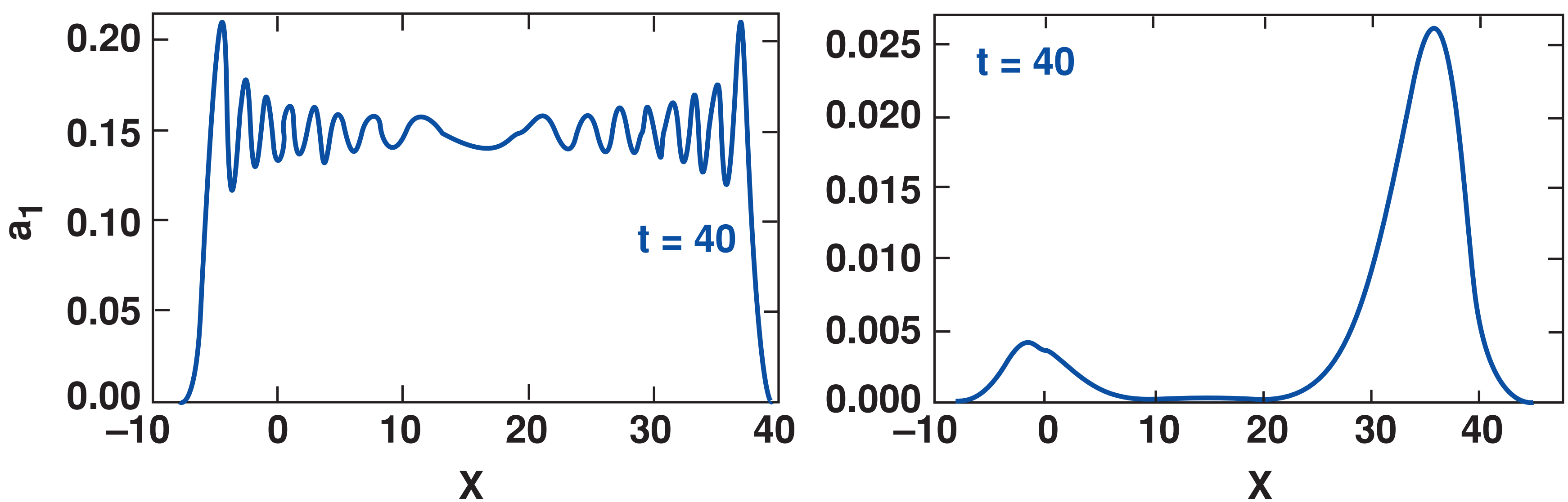
Convective Growth of the Three-Wave Parametric Instability in Nonuniform Plasmas



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Summary

- The well-known theory of three-wave parametric instabilities in an inhomogeneous medium was re-examined. An analytic form was obtained for the Green's function response to an initial delta-function pulse, which allows a more-refined assessment of its properties and greater facility in employing it to model realistic initial conditions.

- Most significantly, we find that physically realistic initial pulses behave quite differently than would be expected on the basis of the singular delta-function result, and more in accord with what would be expected from the theory of convective instabilities in homogeneous plasmas.

Introduction

- Measurements at LLE of two-plasmon wave amplitudes, in regimes where absolute growth would not occur, led us to re-examine the theory of convective growth in nonuniform plasma.
- The classic Rosenbluth solution of the three-wave equations in a nonuniform plasma shows an odd behavior, for a convective instability.
- On the other hand, direct time integration of the equations yielded a result in good accord with the expected behavior of a convective instability.
- We resolve this discrepancy and show that it is due to the use of a nonphysical initial pulse. New analytic expressions are obtained that facilitate accurate and rapid computation and agree with the direct time-integration result.

These are the three-wave parametric instability equations in a nonuniform plasma

$$\frac{\partial a_1}{\partial t} + v_1 \frac{\partial a_1}{\partial x} + \nu_1 a_1 = \gamma_0 a_2 e^{i\kappa' x^2/2}$$

$$\frac{\partial a_2}{\partial t} - v_2 \frac{\partial a_2}{\partial x} + \nu_2 a_2 = \gamma_0 a_1 e^{-i\kappa' x^2/2}$$

It is assumed that κ' is a constant.

$$\kappa' \equiv d(k_0 + k_1 + k_2)/dx$$

The two waves have group velocities, v_1 and v_2, in the opposite directions.

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Rosenbluth obtained solutions to these equations in a classic analysis

- In his famous 1972 paper, he reached the surprising result that even in the case of daughter waves with opposite group velocities, $v_1 v_2 < 0$, the growth could not be absolute in a nonuniform plasma.

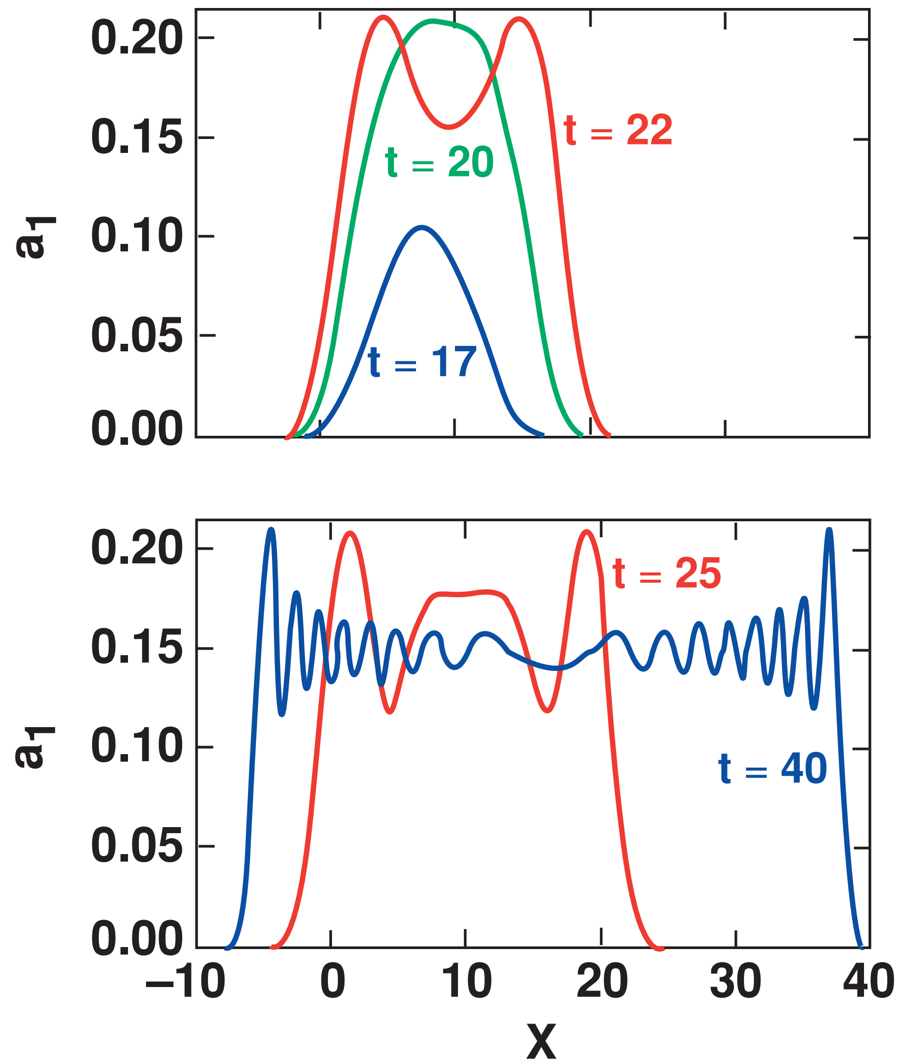
- In a 1973 follow-up paper by Rosenbluth, White, and Liu, an expression was obtained for the amplitude $a_1(x, t)$ resulting from an initial pulse with

$$a_1(x, t = 0) = 0, \quad a_2(x, t = 0) = -\left(\bar{a}/\sqrt{\kappa'}\right)\delta\left(x - x_0\right)$$

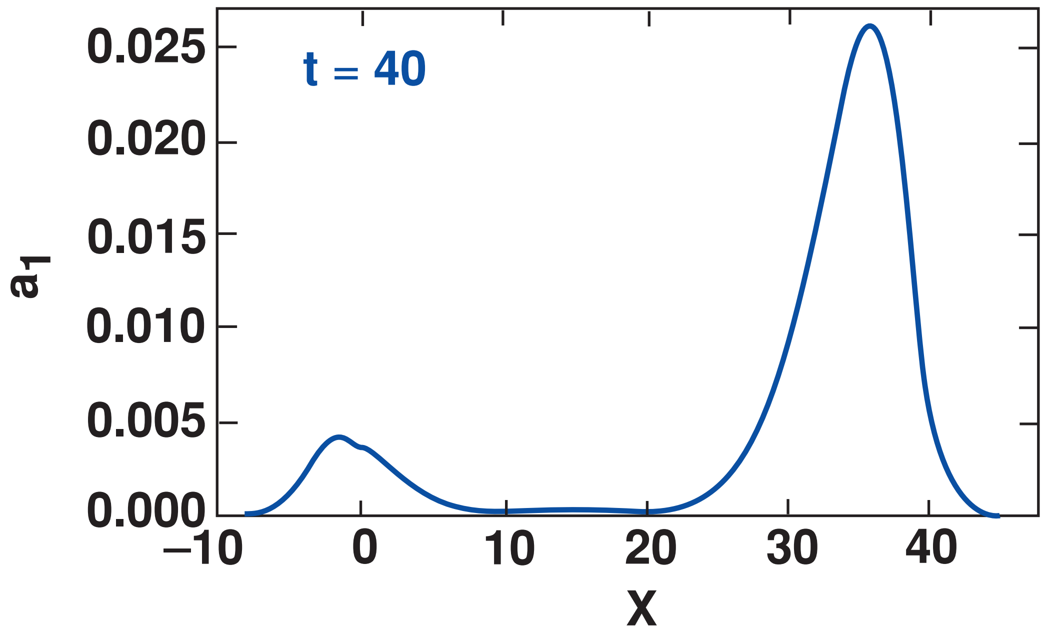
- The final expression for a_1 included an integral over a rapidly varying function and was evaluated numerically for some specific values of v_1, v_2, γ_0 and κ'.

Figure one of the RWL paper has been recalculated, a bit more accurately, for the collisionless case

- The ordinate is normalized to exp($\pi\lambda$).
- "The pulse broadens as it grows and eventually assumes the form of a totally amplified region flanked by two shock fronts."
- This is not the usual picture of a convective instability



Our direct time integration of the three-wave equations yielded a bit of a surprise



- This is for the same parameters as in the previous plot, except that the initial pulse is a narrow Gaussian since a delta pulse cannot be realized numerically (or physically). Here b = 2, where

$$a_2(x, t = 0) = \frac{1}{\sqrt{\pi}\, b} \exp - \left(x^2/b^2\right), \quad a_1(x, t = 0) = 0.$$

- The behavior is much more in accord with the picture of a convective instability in a uniform medium.

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To resolve this discrepancy, the RWL result was used as a Green's function

$$a_{pulse}(x,t) = \frac{1}{\sqrt{\pi}\,b} \int_{-\infty}^{\infty} a_{RWL}\,(x, x_0, t)\, e^{-x_0^2/b^2} dx_0$$

- The calculation is greatly speeded and made more accurate by noting that one can obtain an analytic expression for the integral in the RWL expression for a_1, and it is also valid for all values of x.
- This avoids the need to carry out a double integral over a rapidly oscillating function.
- Excellent agreement with the direct time-integration result was obtained.

Our analytic solution for $a_1(x,t)$ for an initial delta-pulse in a nonuniform plasma

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- It is proportional to a Laguerre polynomial of complex index and imaginary argument.

$$a_1(x,t) = -\frac{\sqrt{\lambda}\,\bar{a}\,\sqrt{v_1 v_2}}{(v_1+v_2)}\, e^{i\kappa' \frac{x_0^2}{4}}\, e^{-\frac{i}{2}\frac{\kappa'(v_1 v_2)^2}{(v_1+v_2)^2}\left(t-\frac{x-x_0}{v_1}\right)^2 - i\frac{\kappa' v_1 v_2}{v_1+v_2}\left(t-\frac{x-x_0}{v_1}\right)x}$$

$$\times\, L_{i\lambda-1}\left[i\kappa'\left(\frac{v_1 v_2}{v_1+v_2}\right)^2\left(t-\frac{x-x_0}{v_1}\right)\left(t+\frac{x-x_0}{v_2}\right)\right]$$

- Valid for $v_1 t > x - x_0 > -v_2 t$, and 0 for x outside this range.

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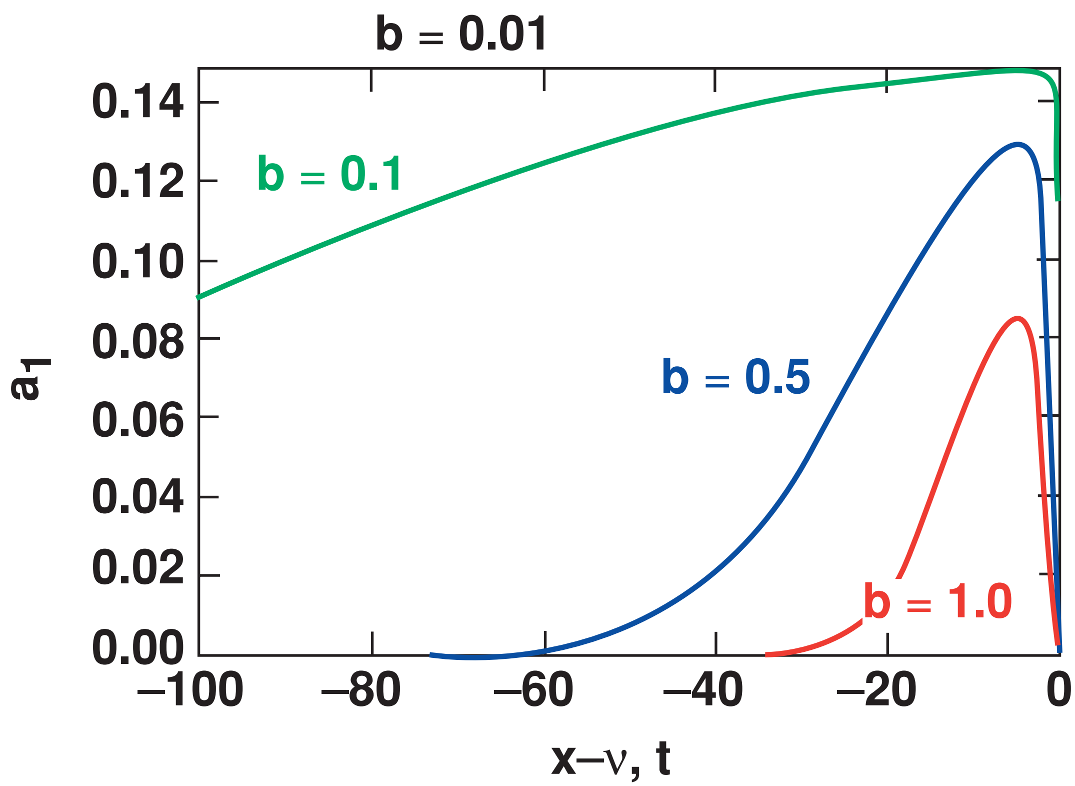
A number of other interesting and useful analytical results were obtained

- We derive an analytical expression for the shape of the convecting pulse (for t, $x \to \infty$). This has a form of a parabolic cylinder function and agrees with our computations.

- In the limit, $b \to 0$, the pulse broadens and goes over to the RWL "plateau."



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An analytical solution for $a_1(x, t)$ in a uniform plasma

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- We rediscovered an (ancient) analytical expression for the amplitude of the spreading wave resulting from delta-pulse in a uniform medium (Bobroff and Haus, 1967). While the instability is absolute for $v_1 v_2 < 0$ and convective for $v_1 v_2 > 0$, it is interesting that the resultant "shock front" height is exactly the same as in the inhomogeneous case.

$$a_{1H}(x,t) = -\frac{\gamma_0 \bar{a}_H}{v_1 + v_2} I_0\left[\frac{2\gamma_0\sqrt{v_1 v_2}}{v_1 + v_2}\sqrt{\left(t - \frac{x - x_0}{v_1}\right)\left(t + \frac{x - x_0}{v_2}\right)}\right]$$

This is valid for $v_1 t > x - x_0 > -v_2 t$, and 0 for x outside this range.

Bobroff and Haus, J. Appl. Phys. 38, 190 (1967).

Summary/Conclusions

- The well-known theory of three-wave parametric instabilities in an inhomogeneous medium was re-examined. An analytic form was obtained for the Green's function response to an initial delta-function pulse, which allows a more-refined assessment of its properties and greater facility in employing it to model realistic initial conditions.

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