High-β Tokamak Equilibria with Poloidal Flows Exceeding the Poloidal Alfvén Velocity

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Motivations

- Fast plasma rotation may improve microscopic and macroscopic tokamak stability.
- In the present work we focus on MHD equilibria with fast poloidal rotation, which could be relevant to the Electric Tokamak (ET).
- Both theory and numerical solutions show that equilibria with super-Alfvénic poloidal flow will show inverted Shafranov shift and outward-positioned separatrix.

Basic equations

- **Continuity:**

$$\nabla \bullet (\rho \vec{v}) = \mathbf{0} \qquad (1)$$

- **Momentum:**

$$\rho \vec{v} \bullet \nabla \vec{v} = \vec{\mathbf{J}} \times \vec{\mathbf{B}} - \nabla \bullet \overleftrightarrow{\mathbf{P}} \qquad \nabla \times \vec{\mathbf{B}} = \mu_{\mathbf{0}} \vec{\mathbf{J}} \qquad (2)$$

$$\overleftrightarrow{\mathbf{P}} \equiv \mathbf{p}_{\perp} \overleftrightarrow{\mathbf{I}} + \Delta \vec{\mathbf{B}} \vec{\mathbf{B}} \qquad \Delta \equiv \left(\mathbf{p}_{||} - \mathbf{p}_{\perp}\right) / \mathbf{B}^{2} \qquad (3)$$

- **Maxwell equations:**

$$\nabla \times \left(\vec{v} \times \vec{\mathbf{B}}\right) = \mathbf{0} \qquad \nabla \bullet \vec{\mathbf{B}} = \mathbf{0} \qquad (4)$$

The system of equations can be reduced to a "Bernoulli" and a "Grad–Shafranov" equation

- **Faraday's law yields the plasma flow:**

$$\vec{v} = M_{A\theta}\vec{v}_A + R\Omega(\Psi)\hat{e}_\varphi \qquad M_{A\theta} = \Phi(\Psi)/\sqrt{\rho} \qquad (5)$$

- **The $\hat{\varphi}$-component of the momentum equation yields the toroidal component of the magnetic field:**

$$B_\varphi R = \frac{I(\Psi) - R^2 M_{A\theta}\sqrt{\rho}\,\Omega(\Psi)}{1 - M_{A\theta}^2 - \Delta} \qquad (6)$$

- **The $\vec{B}$-component of the momentum equation reduces to a "Bernoulli-like" equation for the fluid energy parallel to the field lines:**

$$\frac{1}{2}\frac{(M_{A\theta}B)^2}{\rho} - \frac{1}{2}[R\Omega(\Psi)]^2 + W = H(\Psi) \qquad (7)$$

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*E. Hameiri, Phys. Fluids 26, 230 (1983).

The modified Grad–Shafranov equation depends on six free functions of Ψ

- **The $\nabla\Psi$-component of the momentum equation gives a "GS-like" equation:**

$$\nabla \cdot \left[\left(1 - M_{A\theta}^2 - \Delta\right)\left(\frac{\nabla\Psi}{R^2}\right)\right]$$

$$= -\frac{\partial p_{||}}{\partial\Psi} - \frac{B_\phi}{R}\frac{dI(\Psi)}{d\Psi} - \vec{v}\cdot\vec{B}\frac{d\Phi(\Psi)}{d\Psi} - R\rho\vec{v}_\phi\frac{d\Omega(\Psi)}{d\Psi} - \rho\frac{dH(\Psi)}{d\Psi} + \rho\frac{\partial W}{\partial\Psi}. \qquad \textbf{(8)}$$

- $W(\rho,B,\Psi)$ **is the enthalpy of the plasma and its definition depends on the description of the plasma (MHD or kinetic).**

- $I(\Psi)$, $\Phi(\Psi)$, $\Omega(\Psi)$, $H(\Psi)$, $(\partial p_{||}/\partial\Psi)$, **and** $(\partial W/\partial\Psi)$ **are free functions of** Ψ**.**

***R. Iacono *et al.*, Phys. Fluids B 2, 1794 (1990).**

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The analytic model has an exact solution in the high-flow regime

- **Assuming an external vertical field is assigned outside the plasma and setting $p_1 \sim \Psi$ and $B_{\varphi 2} \sim \Psi$, the GS equation yields**

$$\Psi = -\frac{a^2 B_0}{q_*}\left[\left(\rho^2 - 1\right) + \frac{\nu}{\mu}\left(\rho^3 - \rho\right)\cos(\theta)\right] \quad \textbf{in the plasma} \qquad (10)$$

$$\Psi = -\frac{a^2 B_0}{q_*}\left[\ln(\rho) + \frac{\nu}{2\mu}\left(\rho - \frac{1}{\rho}\right)\cos(\theta)\right] \quad \textbf{in the vacuum} \qquad (11)$$

with $q_* \equiv \dfrac{\pi a^2 B_0}{R_o I}, \quad \nu \equiv \dfrac{\beta_t q_*^2}{\varepsilon}, \quad \mu \equiv 1 - M_{A\theta}^2, \quad \rho \equiv \dfrac{r}{a}.$

- **The shape of the equilibrium depends only on ν/μ and on the normalized minor radius.**

Shafranov shift and β limits can be computed as functions of the poloidal Mach number

- **For** $M_{A\theta} > 1$ **one finds**
  - **the Shafranov shift:**

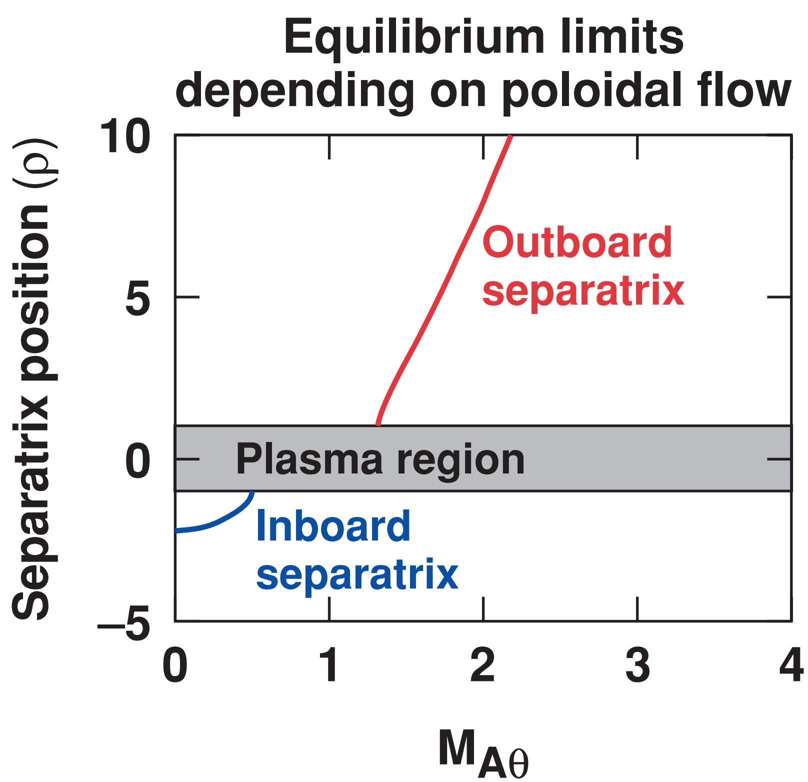
$$\rho_\Delta = \frac{1-\sqrt{1+3(\nu/\mu)^2}}{3(\nu/\mu)}$$

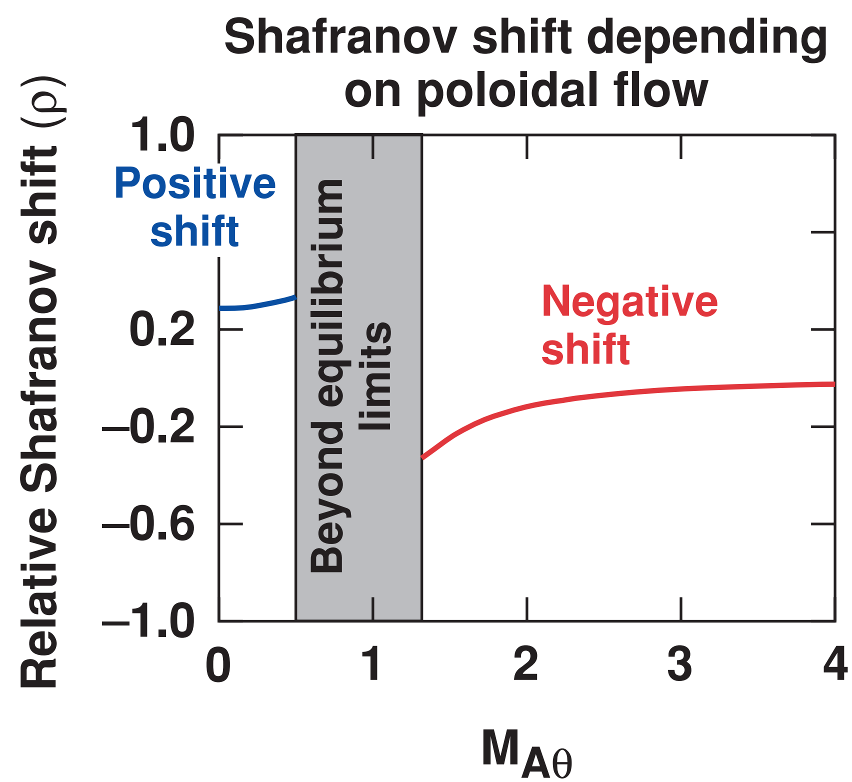
  - **the location of the separatrix (due to the vertical field):**

$$\rho_S = \frac{-1-\sqrt{1-(\nu/\mu)^2}}{(\nu/\mu)}$$

The separatrix is outward and the Shafranov shift is inward for super-Alfvénic poloidal flows

- For a fixed ν, the Shafranov shift and the position of the separatrix (due to the vertical field) are shown.





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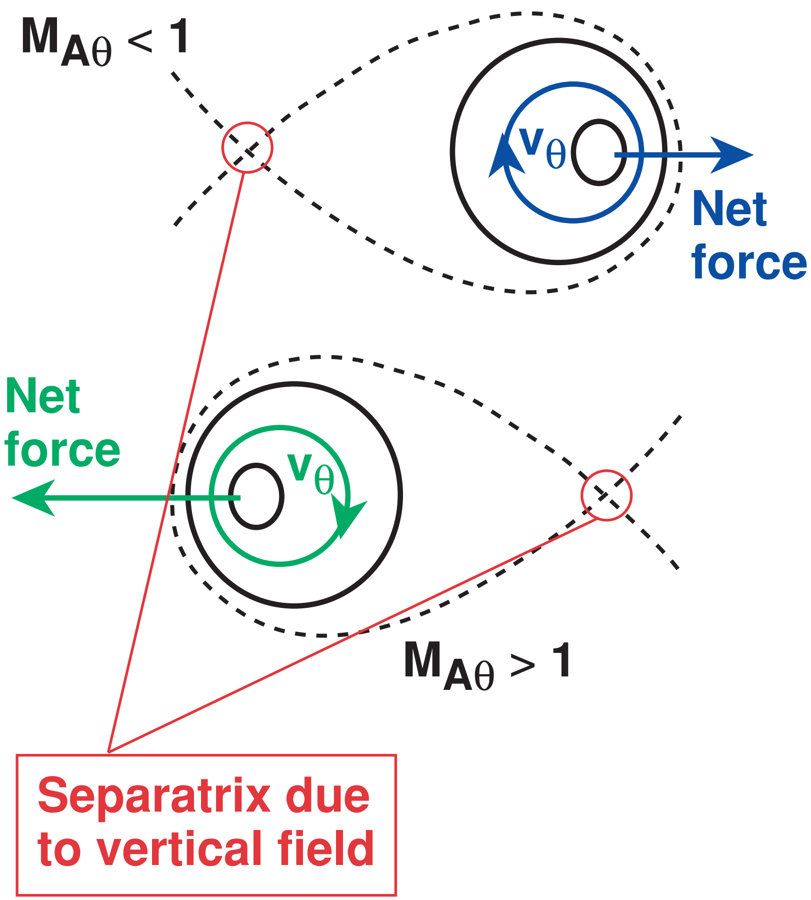
Physical explanation for the inverted Shafranov shift



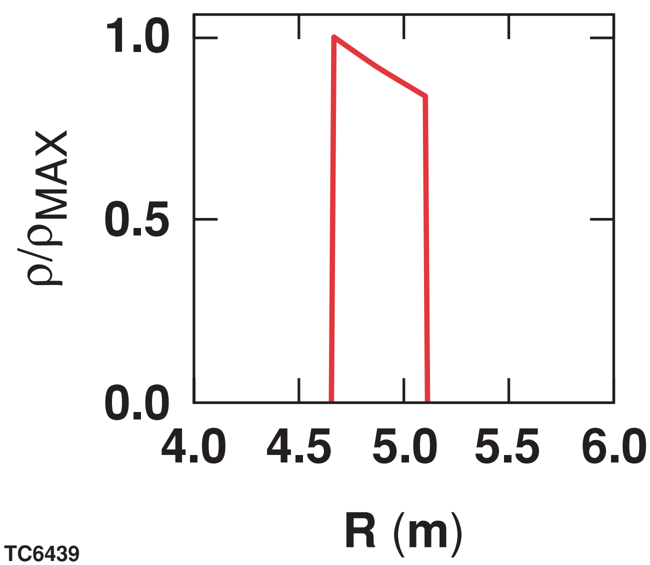
- Plasma rotation produces a centrifugal force, increasing with flow velocity, directed away from the magnetic axis.
- The magnetic surfaces create an ideal “magnetic duct,” which causes the flow velocity to change along the poloidal angle, depending on the Shafranov shift.
- If the Shafranov shift is directed outward, the flow will create a net outward force.
- If the Shafranov shift is directed inward, the net resulting force due to rotation will point inward, and help in balancing the outward-pressure forces.

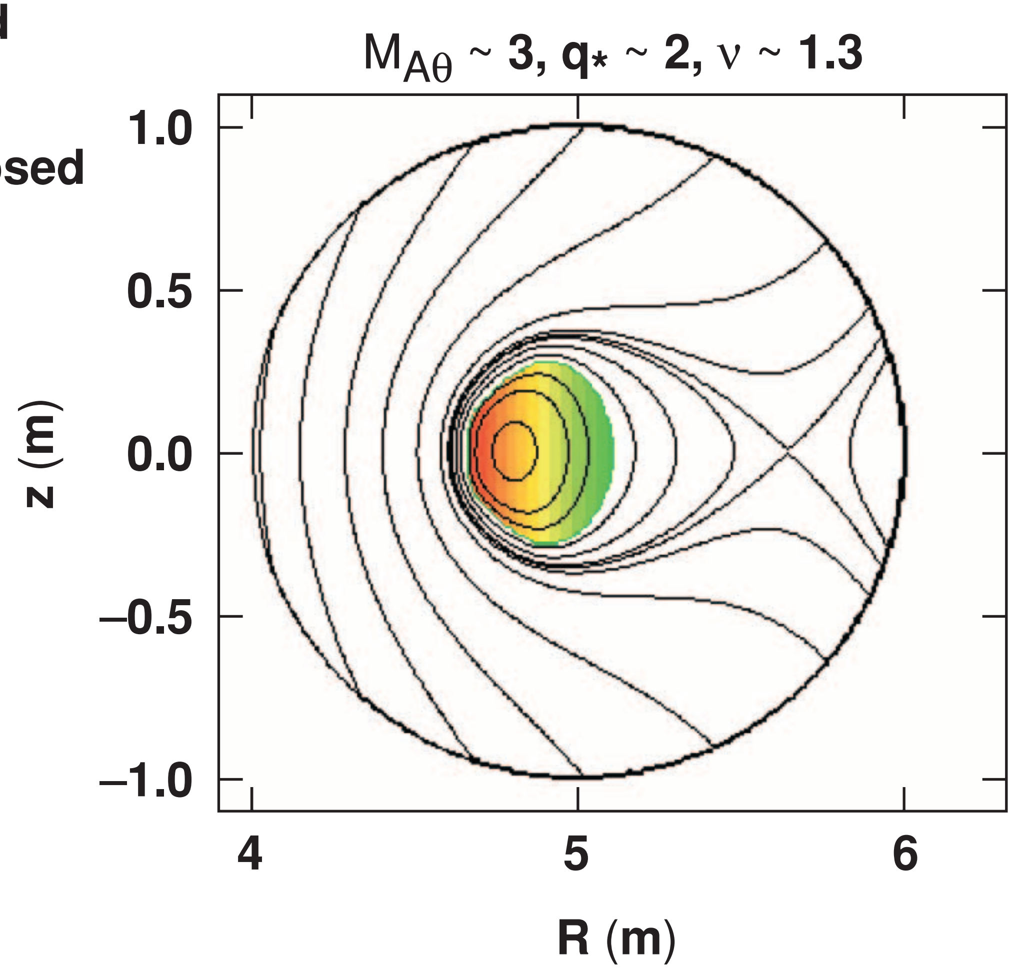
The code *FLOW* can include the vacuum region and solve the equilibrium in the super-Alfvénic regime

- A numerical solution in the vacuum region is required to locate the separatrix.
- The outer vertical field determines the boundary condition.
- Free functions in the plasma are defined to
  - minimize the toroidal rotation
  - assign a roughly constant $M_{A\theta}$
  - assure the plasma is isotropic

FLOW confirms the existence of inward shifted equilibria for $M_{A\theta} > 1$

- Several equilibria, with and without X points in the computational domain, have been computed with *FLOW*.
- Density and Ψ are superimposed in the picture.





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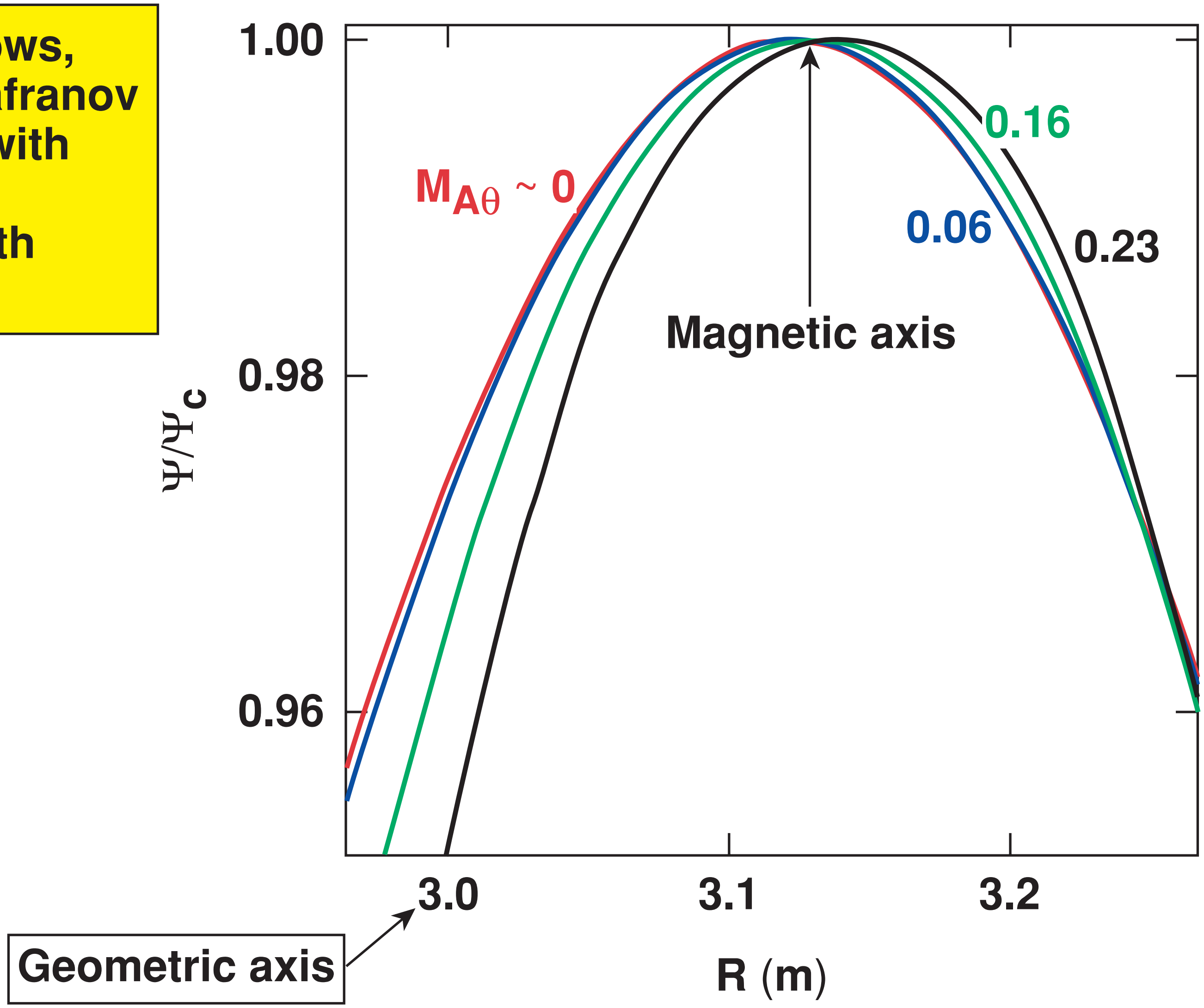
Theory verification

- A qualitative check of the theory has been performed with varying flow.
- For numerical reasons, it is impractical to study the separatrix position as a function of the flow.
- A qualitative comparison is carried out between analytic and numerical Shafranov shift.
- The allowed range for poloidal flow is restricted to the elliptic regions and by the condition of solvability of the Bernoulli equation.

Subsonic flows: *FLOW* shows a small outward Shafranov shift for increasing flows

- **For subsonic flows, the outward Shafranov shift increases with increasing flow, in agreement with analytic theory.**

- **There is no equilibrium for $M_{A\theta} \gtrsim 0.23$.**



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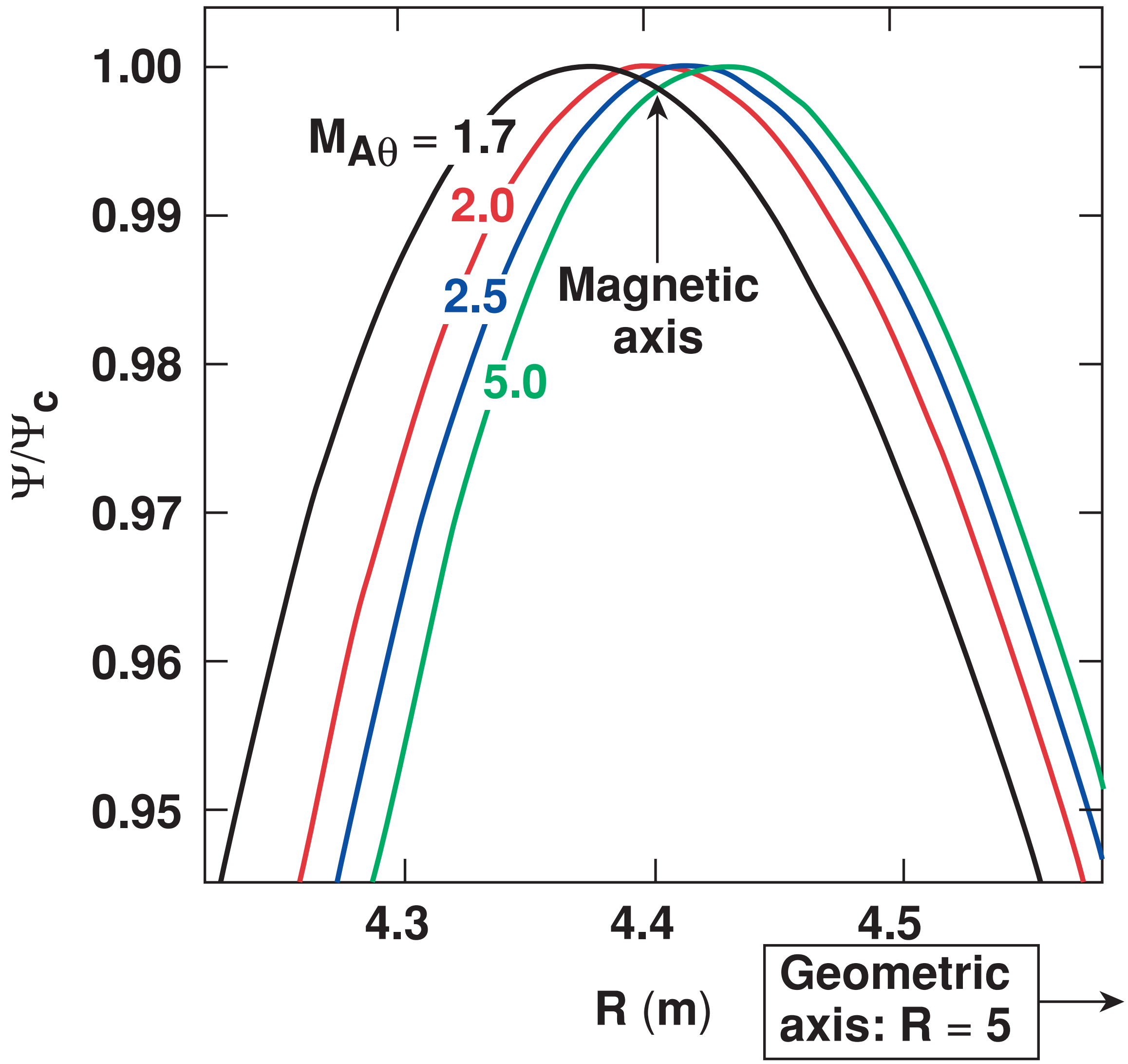
Super-Alfvénic flows: *FLOW* shows a large inward Shafranov shift that decreases as the flow increases

- The inward Shafranov shift decreases with increasing flow, in agreement with theory.
- The shift variation is more pronounced for slower flows, in agreement with theory.

- There is no equilibrium for $M_{A\theta} \gtrsim 1.7$.



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Conclusions

- Equilibria with very fast poloidal rotation could be relevant to the Electric Tokamak.
- MHD equilibria with super-Alfvénic poloidal flows exhibit inverted Shafranov shift and outward-positioned separatrix.
- An analytic equilibrium can be found for $\beta \sim \varepsilon$ and $M_{A\theta} \sim$ constant $>$ 1.
- The code *FLOW* confirms the existence of such super-Alfvénic equilibria.

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