Propagation of Plasma Waves In Weakly Collisional Plasmas

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Wave propagation in weakly collisional plasmas is a topic of continuing interest and controversy. Lenard and Bernstein[1] solved a simplified Fokker–Planck equation exactly, but this result is expressed in terms of an integral that can be difficult to evaluate. Thus more recent investigators have employed approximations such as boundary layer methods[2] and expansion in Hermite polynomials.[3] Su and Oberman[2] found a damping exponent for the perturbed distribution function proportional to the cube of time and/or distance, a result that has been disputed on theoretical and experimental grounds.[3] We derive an analytic solution that is readily evaluated numerically. Roots of the resulting dispersion relation are found and agree with Ref. 3, and the detailed spatial and temporal evolution of the distribution function is shown. This work was supported by the U.S. Department of Energy Office of Inertial Confinement Fusion under Cooperative Agreement No. DE-FC03-92SF19460.

1. A. Lenard and I. B. Bernstein, Phys. Rev. **112**, 1456 (1958).
2. C. H. Su and C. Oberman, Phys. Rev. Lett. **20**, 427 (1968).
3. C. S. Ng, A. Bhattacharjee, and F. Skiff, Phys. Rev. Lett. **83**, 1974 (1999).

Abstract

Wave propagation in weakly collisional plasmas is a topic of continuing interest and controversy. Lenard and Bernstein[1] solved a simplified Fokker–Planck equation exactly, but this result is expressed in terms of an integral that can be difficult to evaluate. Thus, more recent investigators have employed approximations such as boundary layer methods[2] and expansion in Hermite polynomials.[3] Su and Oberman[2] found a damping exponent for the perturbed distribution function proportional to the cube of time and/or distance, a result that has been disputed on theoretical and experimental grounds.[3] We derive an analytic solution that is readily evaluated numerically. Roots of the resulting dispersion relation are found and agree with Ref. 3, and the detailed spatial and temporal evolution of the distribution function is shown.

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Summary

A simplified, linearized Fokker–Planck equation can be solved exactly in terms of gamma functions

- This enables the effects of weak collisionality to be studied easily, both analytically and numerically.
- The discrete spectrum of the Vlasov equation (collective modes) is modified continuously by weak collisionality.
- The continuous spectrum of the Vlasov equation (Case–Van Kampen or “ballistic” modes) is replaced by supralinear spatial and/or temporal decay of initial distribution function perturbations.

Outline

- Adding weak collisionality to the Vlasov equation:
  a simplified Fokker–Planck equation.
- Temporal modes and dispersion relation
- Antenna problem: spatial modes and dispersion relation
- Summary and conclusions

Weak collisionality may be modeled by adding a simplified Fokker–Planck term to the linearized Vlasov equation

- The equation for the perturbed distribution function is

$$\frac{\partial f}{\partial t} + \nu \frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial f}{\partial \nu} E = \nu \frac{\partial}{\partial \nu}\left(\nu f + \nu_t^2 \frac{\partial f}{\partial \nu} \right).$$

- The collision term conserves particles and satisfies the H-theorem but gives a velocity-independent collision frequency.

- The electric field is determined by Poisson's equation:

$$\frac{\partial E}{\partial x} = -4\pi e \int_{-\infty}^{\infty} f(x, \nu, t)\, d\nu .$$

- Look for normal mode solutions of the form $f(x, \nu, t) = \tilde{f}(k, \nu, \omega)\, e^{i(kx - \omega t)}$.

The Vlasov and Poisson equations are combined and converted to dimensionless form

- Following the notation of Ng *et al*. we define

$$u \equiv \nu\big/\left(\sqrt{2}\,\nu_t\right),\ \Omega \equiv \omega\big/\left(\sqrt{2}\,k\nu_t\right),\ \ g \equiv \sqrt{2}\,\nu_t\,\tilde{f}\big/n_0\,,$$

$$g_0(u) \equiv e^{-u^2}\big/\sqrt{\pi},\ \eta(u) \equiv \frac{\alpha}{2}\frac{\partial g_0}{\partial u},\ \alpha \equiv 1\big/k^2\lambda_D^2,\ \text{and}\ \mu \equiv \nu\big/\left(\sqrt{2}\,k\nu_t\right).$$

- The combined Vlasov and Poisson equations then become

$$(u - \Omega)\,g(u) - \eta(u)\int_{-\infty}^{\infty} g(u')\,du' = -i\mu\frac{\partial}{\partial u}\left(ug + \frac{1}{2}\frac{\partial g}{\partial u}\right).$$

- Fourier transforming in velocity with

$$G(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iwu}\,g(u)$$

gives a first-order equation for the transform of the distribution function:

$$\left(-i\frac{d}{dw} - \Omega\right)G(w) + \frac{i\alpha}{2}we^{-\frac{w^2}{4}}G(0) = i\mu\left[w\frac{d}{dw}G(w) + \frac{1}{2}w^2G(w)\right].$$

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The solution can be expressed in terms of the incomplete gamma function

- The dispersion relation is

$$1 = -\alpha\left\{1 + \frac{i\Omega}{\mu}\left(2\mu^2\right)^{\left[\left(1/2\mu^2\right)-\left(i\Omega/\mu\right)\right]} e^{1/2\mu^2}\,\gamma\left(\frac{1}{2\mu^2} - \frac{i\Omega}{\mu}, \frac{1}{2\mu^2}\right)\right\}.$$

- For small values of μ, numerical evaluation is facilitated by writing the dispersion relation as

$$1 = \alpha\left\{\frac{i\Omega}{\mu}\left[G\left(\frac{1}{2\mu^2} - \frac{i\Omega}{\mu}, \frac{1}{2\mu^2}\right)\right] - \left(2\mu^2\right)^{\left[\left(1/2\mu^2\right)-\left(i\Omega/\mu\right)\right]} e^{1/2\mu^2}\,\Gamma\left(\frac{1}{2\mu^2} - \frac{i\Omega}{\mu}\right) - 1\right\},$$

where $G(a,x) \equiv x^{-a} e^{x}\Gamma(a,x) = \cfrac{1}{x + \cfrac{1-a}{1 + \cfrac{1}{x + \cfrac{2-a}{1 + \cfrac{2}{x + \cfrac{3-a}{1 + \ldots}}}}}}$.

To study behavior as $\mu \to 0$ it is useful to expand the dispersion relation in μ

- To first order in μ the dispersion relation becomes

$$1 = -\alpha\left\{1 + i\sqrt{\pi}\,\Omega e^{-\Omega^2}\text{erfc}(-i\Omega) + \frac{i\Omega}{3}\left[2\left(1-\Omega^2\right) + i\sqrt{\pi}\,\Omega e^{-\Omega^2}\left(3-2\Omega^2\right)\text{erfc}(-i\Omega)\right]\mu\right\}.$$

- When $\mu = 0$, this reduces to the familiar Landau dispersion relation for the collective modes of the plasma.

- For $\mu \neq 0$ a discrete set of roots approach the Landau roots as $\mu \to 0$, but the Case–Van Kampen continuum is not recovered.

Experiments usually involve spatial propagation from an antenna rather than temporal decay

- **Again, start with the Fokker–Planck and Poisson equations:**

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial f_0}{\partial v}E = \nu\frac{\partial}{\partial v}\left(vf + v_t^2\frac{\partial f}{\partial v}\right)$$

$$\frac{\partial}{\partial x}\left(E - E_{ext}\right) = -4\pi e\int_{-\infty}^{\infty} f dv,$$

where the antenna source is $E_{ext}(x, t) = E_e\,\delta(x)\,e^{-i\omega t}$.

- **Let** $f = \frac{n_0}{\sqrt{2}v_t}g, v = \sqrt{2}v_t u,\ \nu = \omega\mu,\ k = \frac{\omega}{\sqrt{2}v_t}\kappa,$

and $\eta(u) = \frac{\omega_p^2}{\omega^2}\frac{\partial g_0}{\partial u}$:

$$\left(u - \frac{1}{\kappa}\right)g - \frac{\eta(u)}{\kappa^2}\int_{-\infty}^{\infty} g(u')\,du' + \frac{i}{\sqrt{2\pi}}\frac{u_e}{\kappa}\frac{\omega^2}{\omega_p^2}\eta(u) = -i\frac{\mu}{\kappa}\frac{\partial}{\partial u}\left(ug + \frac{1}{2}\frac{\partial g}{\partial u}\right).$$

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The solution for the perturbed distribution function is again expressed in terms of gamma functions

- Define $d(a,x) \equiv e^{x} x^{-a} \gamma(a,x)$; this is a single-valued analytic function in the finite complex *a* and *x* planes, except for simple poles when *a* is a non-positive integer.

- Fourier transforming the perturbed distribution in velocity

$$G(\kappa,w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\kappa,u)\, e^{-iwu} du,$$

the solution is

$$G(\kappa,w) = \frac{\dfrac{iu_e}{\pi\kappa} e^{-\frac{w^2}{4}} \left\{ 1 + \dfrac{i}{\mu} d\left[\dfrac{\kappa^2}{2\mu^2} - \dfrac{i}{\mu}, \dfrac{\kappa^2}{2\mu^2} \left(1 - \dfrac{\mu}{\kappa} w \right) \right] \right\}}{1 + \dfrac{1}{\kappa^2} \dfrac{2\omega_p^2}{\omega^2} \left[1 + \dfrac{i}{\mu} d\left(\dfrac{\kappa^2}{2\mu^2} - \dfrac{i}{\mu}, \dfrac{\kappa^2}{2\mu^2} \right) \right]}.$$

The denominator gives the collective modes; the numerator gives the "free-streaming" modes

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- The dispersion relation is

$$1+\frac{1}{\kappa^2}\frac{2\omega_p^2}{\omega^2}\left[1+\frac{i}{\mu}d\left(\frac{\kappa^2}{2\mu^2}-\frac{1}{\mu},\frac{\kappa^2}{2\mu^2}\right)\right]=0.$$

- As $\mu \to 0$ the roots approach the Landau collective modes of the Vlasov equation.

- The numerator gives the spatial decay of the "free streaming" part of the perturbed distribution function after the collective modes have Landau damped away. In dimensional variables the result is

$$g(\nu,x)=-\frac{2U_E}{\sqrt{\pi}}e^{-\frac{\nu^2}{2\nu_t^2}}e^{i\omega\frac{x}{\nu}}e^{-\frac{\nu}{3}\omega^2\nu_t^2\frac{x^3}{\nu^5}}\text{ for }\frac{x}{\nu}>0,\ 0\text{ for }\frac{x}{\nu}<0.$$

Summary/Conclusions

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