Landau Damping and Transit-Time Damping of Localized Plasma Waves in General Geometries

The collisionless damping of electrostatic plasma waves, first predicted by Landau[1] in 1946 and since rederived in many ways and confirmed experimentally, has become perhaps the most well known phenomenon in plasma physics. Landau damping plays a significant practical role in many plasma experiments and applications but has continued to be of great interest to theorists as well. Much of this interest stems from the counterintuitive nature of the result itself (that waves carrying free energy dissipate with no increase in entropy) coupled with the rather abstruse mathematical nature of Landau's original derivation. For these reasons there was even some controversy over the reality of the phenomenon,[2] until it was actually observed in experiments.

The usual derivation of Landau damping[3] begins by linearizing the Vlasov equation for an infinite homogeneous collisionless plasma. The linearized Vlasov equation is Fourier transformed in space and Laplace transformed in time, and the resulting equations in transform space are then solved algebraically to yield a relation between the perturbing field and the perturbed distribution function. Alternatively, this relation may be obtained by directly integrating the linearized Vlasov equation in configuration space using the method of characteristics,[4] also known as "integration over unperturbed orbits," and then performing the Fourier and Laplace transforms. Integration of this relation over particle velocities then leads to the dielectric response function and a dispersion relation for the plasma waves. Performing the integration over velocities entails the avoidance of a pole on the real axis by deforming the integration contour into the complex velocity plane. (Details can be found in most introductory plasma physics texts.) While this derivation is mathematically elegant, it is physically rather obscure, especially in regard to the introduction of complex velocities. For this reason, many "physical" derivations of Landau damping have been published, employing only real physical quantities throughout.[5,6] In these derivations, the energy transferred from the wave to each particle is calculated directly and then integrated over the particle distribution function to give the damping. In these physical derivations, however, the perturbed particle orbit must be determined and the wave-particle energy transfer calculated to second order in the field amplitudes. (The reasons for this will be discussed below.) Calculation of the perturbed particle orbit in a time-varying field is rather complicated, even for a plane wave, involving as it does iterated time integrals of the equation of motion. Such complications are contrary to the motivation for a physical derivation of Landau damping, which is to provide a simple, physically intuitive explanation of the phenomenon. Furthermore, they ought to be unnecessary since the transform derivation requires only unperturbed orbits and first-order quantities. One of the results that will emerge below is a physical derivation of Landau damping based solely on unperturbed orbits.

Strictly speaking, the term "Landau damping" applies only to the damping of infinite plane waves in homogeneous plasmas. Localized electrostatic perturbations in inhomogeneous plasmas, however, are also damped by collisionless processes.[7] Particles transiting the region containing the wave exchange energy with it; for a thermal distribution of particles, this results in a net transfer of energy from the wave to the particles and a consequent damping of the wave. This process is usually referred to as "transit-time damping."[8,9] Since, in general, the Fourier transform method used by Landau is difficult to apply in inhomogeneous plasmas, transit-time damping calculations employ the physical approach described above: the energy transferred to each particle is calculated and then integrated over the particle distribution function. Again, however, this requires that the perturbed particle orbits be determined and the energy transfer be calculated to second order in the fields; for a localized field in an inhomogeneous plasma, this is much more complicated than for a plane wave. Since Landau damping can be calculated based solely on the unperturbed orbits, it is natural to inquire if transit-time damping could also be calculated without invoking the perturbed orbits. One of the main purposes of this article is to show how this can be done.

First, we give a physical derivation of transit-time damping in a plasma slab of finite width based on unperturbed orbits and investigate how the damping of a plasma wave confined to the slab varies with slab width and mode number. We also show that the result reduces to the usual Landau-damping expression as the width becomes large. Next, we present a similar analysis for spherical geometry followed by a brief discussion of the cylindrical case, which is covered in more detail in a future article.[10] Finally, in an appendix, we show formally that in general geometries our approach gives results equivalent to those obtained by other methods that require the use of perturbed orbits and higher-order terms.

Transit-Time Damping in Slab Geometry

Our approach to transit-time damping may be outlined as follows: Consider a localized oscillating electrostatic field that may be regarded as stationary in time, i.e., its oscillation amplitude is unchanging. In practice, this may correspond to a situation of weak damping, where the damping rate is much smaller than the oscillation frequency (as is often the case for Landau damping), or to a situation where wave energy lost to damping is replenished by an external source, such as in the case of stimulated Raman or Brillouin scattering, where the electrostatic wave is driven by interaction with an electromagnetic pump wave. We assume that the particle distribution function f_0 depends solely on the particle energy E, and we further assume that collisional damping is negligible and take the plasma to be collisionless, so that $f_0(E)$ satisfies the Vlasov equation. Consider a six-dimensional phase-space volume element dV, which passes through the localization volume in time Δt and emerges as the volume element dV^*. Since the Vlasov equation conserves phase-space volume, we have $\left|dV^*\right| = |dV|$, though the shape of the volume element may change. Through interaction with the field, each particle in dV acquires an energy increment ΔE, which may be positive or negative. Since the situation is stationary and the Vlasov equation is invariant under time reversal, the time-reversed process must be occurring simultaneously. In the reversed process, the volume element dV^* enters the localization volume and emerges as dV, each particle in the volume *losing* the energy increment ΔE in time Δt. The net rate at which energy is transferred to the particles associated with dV is then

$$\Delta P = \left\langle \frac{\Delta E}{\Delta t} \left[f_0(E) dV - f_0(E + \Delta E) dV^* \right] \right\rangle$$

$$\cong - \frac{\left\langle (\Delta E)^2 \right\rangle}{\Delta t} \frac{\partial f_0}{\partial E} dV, \qquad (1)$$

where the angle brackets indicate averaging over the field phase. Integration of this quantity over the phase space within the localized volume then gives *twice* the collisionless power transfer to the electrons since the phase space is effectively included twice in the integration (both forward and backward in time).

To illustrate, we now calculate the average energy gain rate of electrons crossing a one-dimensional slab region containing a standing-wave electrostatic field. We will obtain a simple expression for the field damping rate as a function of the slab length (for fixed oscillation frequency and wavelength).

Consider a standing-wave electrostatic potential, ϕ, of real frequency ω:

$$\phi = -\frac{C}{k} \sin(kx) \cos(\omega t)$$

in the slab region with boundaries at $x = 0$ and $x = L$. Here C is a constant inside the slab and vanishes outside, and $kL = 2\pi j$ with j a positive integer so that the potential is continuous. The corresponding electrostatic field is

$$E(x,t) = C \cos(kx) \cos(\omega t).$$

We also assume that electrons with a constant number density n_0 and a velocity distribution $f_0(E)$ are streaming constantly and freely through this region from the left at $x = 0$ and from the right at $x = L$. The density and temperature are chosen such that $\omega_{\mathrm{pe}}^2 >> 3k^2 v_T^2$, where ω_{pe}^2 is the usual plasma frequency and v_T the thermal velocity, so that weak Landau damping and quasi-steady-state conditions obtain. The frequency ω and wave number k then satisfy the Bohm-Gross dispersion relation $\omega^2 = \omega_{\mathrm{pe}}^2 + 3k^2 v_T^2 \approx \omega_{\mathrm{pe}}^2$. We can also treat the case of stronger damping, with $\omega_{\mathrm{pe}}^2 \sim 3k^2 v_T^2$, if we assume that the steady state of the field is maintained by an external source such as the stimulated Raman scattering instability.

To first order in the field amplitude C, the velocity increment obtained by an electron of initial velocity v crossing the slab is simply

$$\Delta v = -\int_0^T \frac{eC}{m} \cos(kvt) \cos(\omega t + \phi) dt,$$

where we have used the unperturbed orbit $x = vt$. Here ϕ is the phase of the field at the time of entrance of the particle, and $T = L/v$. To this order, the energy change ΔE is given by

$\Delta E = mv\Delta v$. It is a simple matter to carry out the integral and then average $(\Delta E)^2$ over the phase. Note that $kvT = kL = 2\pi j$ and hence $\exp(\pm kvT) = 1$. The result is

$$\left\langle (\Delta E)^2 \right\rangle = \frac{(evC)^2}{2}\sin^2\left(\frac{\omega T}{2}\right)\left[\frac{1}{\omega + kv} + \frac{1}{\omega - kv}\right]^2,$$

and Eq. (1) becomes

$$\Delta P = -\frac{(evC)^2}{2T}\sin^2\left(\frac{\omega T}{2}\right) \times \left[\frac{1}{\omega + kv} + \frac{1}{\omega - kv}\right]^2 n_0 \frac{df_0(v)}{dE}\,dv.$$

The net power transferred is obtained by integrating this expression over the phase space within the slab volume, noting that $T = L/|v|$. The result is

$$P = -\frac{\omega_{\text{pe}}^2 C^2}{16\pi}\int_{-\infty}^{\infty}\sin^2\left(\frac{\omega L}{2|v|}\right) \times \left[\frac{1}{\omega + kv} + \frac{1}{\omega - kv}\right]^2 v|v|\frac{df_0(v)}{dv}\,dv, \tag{2}$$

where we have divided by 2 to compensate for the double-counting of phase space, as noted earlier. Note also that although the familiar resonant denominators appear in the integrand, they do not represent poles because of the sine function, so the difficulties in dealing with poles in the velocity integration that arise in Landau's calculation do not appear here.

The energy damping rate follows by dividing this result by the total plasma-wave energy within the slab volume. This energy is

$$W = \int_0^L \left\langle \frac{\mathrm{E}^2(x,t)}{4\pi} \right\rangle dx = \frac{C^2 L}{16\pi}, \tag{3}$$

where the angle brackets denote averaging over time; hence, the field *amplitude* damping rate is *half* of (2) divided by (3):

$$\gamma = -\omega_{\text{pe}}^2 \int_0^{\infty}\sin^2\left(\frac{\omega L}{2v}\right) \times \left[\frac{1}{\omega + kv} + \frac{1}{\omega - kv}\right]^2 \frac{v^2}{L}\frac{df_0(v)}{dv}\,dv. \tag{4}$$

It is easy to show that this reduces to the Landau value in the infinite slab-length limit. Without loss of generality, we may take ω and k positive. If $v \neq \omega/k$, the integrand is finite and thus gives no contribution to γ as $L \to \infty$ (keeping k fixed, which means increasing L in wavelength steps, or j in integral steps). For $v \to \omega/k$, the integrand varies directly as L and becomes infinite. Clearly, the integrand is proportional to $\delta(v - \omega/k)$ in this limit. Replacing nonresonant values of v by ω/k and defining the integration variable $q \equiv \pi(L/\lambda)(\omega/kv - 1)$, where $\lambda = 2\pi/k$ is the wavelength, yields

$$\gamma = -\omega_{\text{pe}}^2\left(\frac{\omega}{k}\right)^2 \left.\frac{df_0}{dv}\right|_{\frac{\omega}{k}} \frac{1}{2\omega}\int_{-\pi L/\lambda}^{\infty}\frac{\sin^2 q}{q^2}\,dq.$$

In the limit of an infinite homogeneous plasma $L/\lambda \to \infty$, we obtain

$$\gamma = -\frac{\pi\omega_{\text{pe}}^2\omega}{2k^2}\left.\frac{df_0}{dv}\right|_{\frac{\omega}{k}}, \tag{5}$$

which is the familiar Landau damping rate for electrostatic waves in a homogeneous plasma.

Colunga *et al.*[11] have also obtained an expression for transit-time damping in a slab and noted that it can be represented as the Landau damping of the Fourier components of the localized electric field, which also gives (5) as the size of the slab increases. Their derivation, however, requires calculation of the wave-particle energy transfer to second order (i.e., use of perturbed orbits.)

We next investigate the damping rate's dependence on the slab size and plasma parameters. Assuming a Maxwellian distribution for $f_0(E)$ and changing the integration variable to $z \equiv \omega/kv$, Eq. (4) becomes

$$\gamma = \frac{4F^3\omega_{\text{pe}}^2\lambda}{(2\pi)^{3/2}L\omega}\int_0^{\infty}\frac{\sin^2\left(\frac{\pi L}{\lambda}Z\right)\exp\left(-F^2/2z^2\right)}{\left(1-z^2\right)^2 z^3}\,dz. \tag{6}$$

Here, $F = \omega/kv_T$, with v_T the electron thermal velocity. For values of F well above unity, we have $\omega \approx \omega_{\text{pe}}$ and $F \approx (k\lambda_D)^{-1}$. The integral above is readily evaluated, for fixed F, and its variation with j is shown in Figs. 75.37(a) and 75.37(b) for $F = 6$ and $F = 4$, respectively. What is actually plotted is the ratio of γ to γ_{L}, where γ_{L} is the infinite slab limit $(L/\lambda \to \infty)$ of Eq. (6),

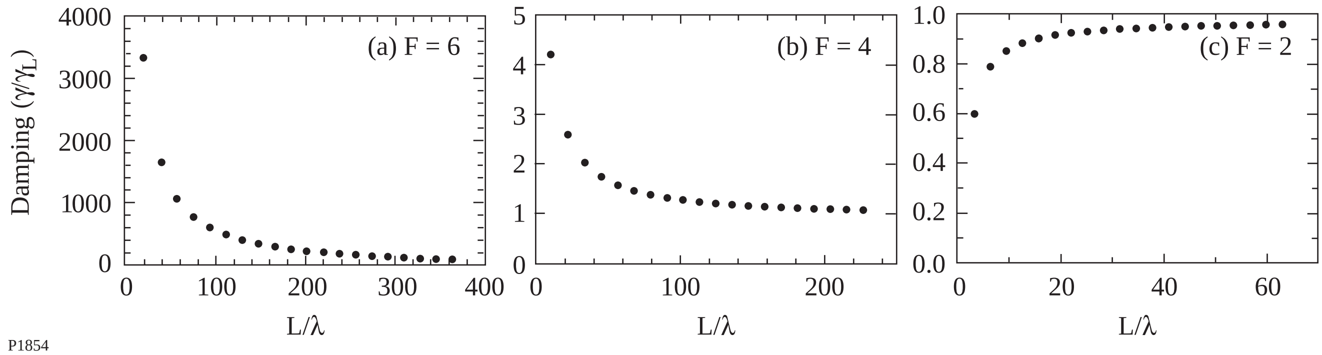

Figure 75.37
Damping rates for a standing plasma wave in a slab of length L, normalized to the Landau damping rate for an infinite homogeneous plasma γ_L. In (a)–(c) results are presented for three values of the parameter $F = \omega/kv_t$, with smaller values of F corresponding to stronger Landau damping.

$$\gamma_L = \left(\frac{\pi}{8}\right)^{1/2} \frac{F^3\omega_{pe}^2}{\omega} \exp\left(\frac{-F^2}{2}\right), \tag{7}$$

the usual Landau damping value. The Landau result arises from the resonant part of the integral; the nonresonant part gives rise to the finite geometry transit-time component of the damping.

Note the monotonic decrease in damping to the usual Landau value as L/λ increases. The value of the ratio at $L/\lambda = 1$ increases as F increases and can be quite large; hence, the transit-time damping can be much larger than the Landau rate for finite slabs. Note, however, that γ_L decreases exponentially with increasing F.

The nonresonant contribution does not always lead to augmentation of the Landau damping rate. As F decreases, the variation with L/λ reverses and the damping *increases* monotonically to the Landau value, as shown in Fig. 75.37(c) for $F = 2$. The general trend seems to be that the finite geometry increases the damping when the infinite geometry (Landau) limit of the damping is small (large F) and reduces damping when the infinite geometry limit is large. An analogous trend appears in the spherical and cylindrical cases, as discussed below, and a qualitative interpretation is presented in the **Conclusions** section.

Transit-Time Damping in Spherical Geometry

As an example of a finite three-dimensional calculation, we now examine the damping of electrostatic modes trapped in a sphere of radius R with a homogeneous internal density n_0. To illustrate the method as simply as possible, we consider only modes with no angular dependence (angular mode numbers $l = m = 0$); more complicated potentials and density profiles will give rise to more complicated forms of the function G, defined in Eq. (10) below, but can be handled by the same basic approach.

The potential inside the sphere is taken to be

$$\phi(r,t) = Aj_0(kr)\cos(\omega t + \alpha), \tag{8}$$

corresponding to a standing spherical wave, where $j_0(x) = \sin x/x$ denotes the spherical Bessel function of order zero, and α is an arbitrary constant representing the phase of the wave, to be averaged over below. The boundary condition is $j_0(kR) = 0$, so k may be any of a discrete set of wave numbers determined by the roots of the Bessel function.

Let $t = 0$ be the time when a particle is closest to the center of the sphere. We obtain its change in energy by integrating over the unperturbed orbit:

$$\Delta E = -e\int_{-t_0}^{t_0} \mathbf{v}\cdot\nabla\phi(\mathbf{r},t)\,dt.$$

Here $2t_0 = \sqrt{R^2 - b^2}/v$ is the time required to cross the sphere, where b is the distance of closest approach to the center of the sphere. The total derivative of the potential is

$$\frac{d}{dt}\phi[\mathbf{r}(t),t] = \mathbf{v}\cdot\nabla\phi[\mathbf{r}(t),t] + \frac{\partial}{\partial t}\phi[\mathbf{r}(t),t],$$

so the above integral can be written

$$\Delta E = -e\int_{-t_0}^{t_0} \left\{\frac{d}{dt}\phi\left[\mathbf{r}(t),t\right] - \frac{\partial}{\partial t}\phi\left[\mathbf{r}(t),t\right]\right\}dt\,.$$

The potential seen by the particle is the same before and after passing through the sphere, so

$$\int_{-t_0}^{t_0} \frac{d}{dt}\phi\left[\mathbf{r}(t),t\right]dt = 0$$

and

$$\Delta E = e\int_{-t_0}^{t_0} \frac{\partial}{\partial t}\phi\left[\mathbf{r}(t),t\right]dt\,.$$

Substituting the form of the potential, changing the integration variable to $s = kvt$, and averaging over the phase α gives

$$\left\langle \Delta E^2 \right\rangle = \frac{\omega^2 e^2 A^2}{2k^2 v^2}\, G^2\left(kR, kb, \frac{\omega}{kv}\right), \qquad (9)$$

where

$$G\left(kR, kb, \frac{\omega}{kv}\right) \equiv \int_{-k\sqrt{R^2-b^2}}^{k\sqrt{R^2-b^2}} \times j_0\left(\sqrt{k^2b^2 + s^2}\right)\cos\left(\frac{\omega s}{kv}\right)ds\,, \qquad (10)$$

a function that must be evaluated numerically.

Next we must integrate Eq. (1), the power loss in an element of phase-space volume, over the six-dimensional phase space inside the sphere. The total power being transferred to particles in the sphere is then

$$P = \int_0^{\pi} d\theta_r \sin\theta_r \int_0^{2\pi} d\phi_r \times \left\{\int_0^R drr^2 \int_0^{\infty} dvv^2 \int_0^{\pi} d\theta_v \int_0^{2\pi} d\phi_v \left[-\frac{1}{2}\frac{\left\langle \Delta E^2 \right\rangle}{\Delta t}\frac{\partial f_0}{\partial E}\right]\right\}, \qquad (11)$$

where the factor 1/2 in the integrand compensates for the double-counting of phase space, as noted earlier in the **Transit-Time Damping** section. Because of the spherical symmetry, the term in braces must be independent of θ_r and ϕ_r, so for convenience we can evaluate it at $\theta_r = \phi_r = 0$ and obtain

$$P = 4\pi n_0 \left\{\int_0^R drr^2 \int_0^{\infty} dvv^2 \int_0^{\pi} d\theta_v \int_0^{2\pi} d\phi_v \times \left[-\frac{1}{2}\frac{\left\langle \Delta E^2 \right\rangle}{\Delta t}\frac{\partial f_0}{\partial E}\right]\right\}\Bigg|_{\theta_r=\phi_r=0}. \qquad (12)$$

For $\theta_r = \phi_r = 0$ we can use the relation $b/r = \sin\theta_v$ to convert the integral over θ_v to an integral over b:

$$\int_0^{\pi} d\theta_v \sin\theta_v \rightarrow 2\int_0^{\pi/2} d\theta_v \sin\theta_v \rightarrow 2\int_0^r db \frac{b/r}{\sqrt{r^2 - b^2}}. \qquad (13)$$

From Eqs. (9) and (10) we see that $\left\langle \Delta E^2 \right\rangle$ is independent of r for fixed b, so using Eq. (13) and $\Delta t = 2\sqrt{R^2 - b^2}/v$, we can perform the r and ϕ_v integrals in Eq. (12):

$$P = \frac{2\pi^2\omega^2 e^2 A^2 n_0}{k^2} \times \int_0^{\infty} dvv\left[-\frac{\partial f_0(E)}{\partial E}\right]\int_0^R dbbG^2\left(kR, kb, \frac{\omega}{kv}\right). \qquad (14)$$

The amplitude-damping rate is now given by $\gamma/\omega = P/2\omega W$, where W is the wave energy contained in the sphere:

$$W = \int_V \left\langle \frac{E^2}{4\pi} \right\rangle_t dV\,.$$

From Eq. (8) we have

$$\left\langle E^2 \right\rangle_t = \frac{1}{2}k^2 A^2 j_0'^2(kr),$$

so

$$W = \int_V \frac{E_{\max}^2}{16\pi}dV = \frac{1}{2}k^2A^2\int_0^R r^2 j_0'^2(kr)\,dr = \frac{1}{4}RA^2\,. \qquad (15)$$

As $R \rightarrow \infty$ with k fixed, the electrostatic wave will locally come to look like a plane wave with wave number k throughout most of the volume of the sphere, so we might expect that in this limit Eq. (14) should give the usual Landau damping rate for such a wave. In Appendix A we show that this is indeed the case.

As in the slab geometry, we can characterize the wave parameters by the quantity $F = \omega/kv_T$ and calculate the damping rates obtained from Eqs. (14) and (15) as the radius of the sphere changes. Figures 75.38(a)–75.38(c) show the results for $F = 6$, 4, and 2, respectively. As in the slab case, we find that the results lie above the Landau limit when the damping is weak (F large), and below when the damping is strong (F small).

Cylindrical Geometry

The case of cylindrical geometry is somewhat more complicated than the slab and spherical geometries because there are two independent components of the wave vector: axial and radial. The cylindrical case is analyzed in detail in a forthcoming article,[10] where the results are applied to the problem of stimulated Raman scattering in a self-focused light filament in a laser-produced plasma. Here we merely note that the damping rate can be shown both analytically and numerically to approach the Landau value as the radius becomes large, and we show some results for the case of a purely radial wave vector for the same values of $F = \omega/kv_T$ as in the slab and spherical cases [Figs. 75.39(a)–75.39(c)]. Once again, we find that the finite radius results lie below the Landau value for F small and above for F large.

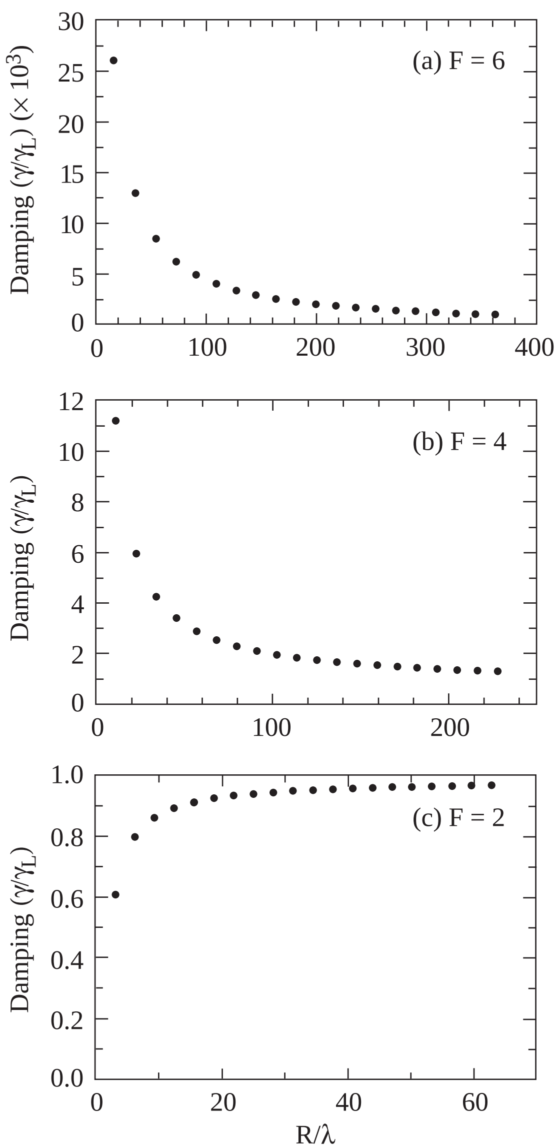

Figure 75.38
Same as Fig. 75.37, but for a sphere of radius R.

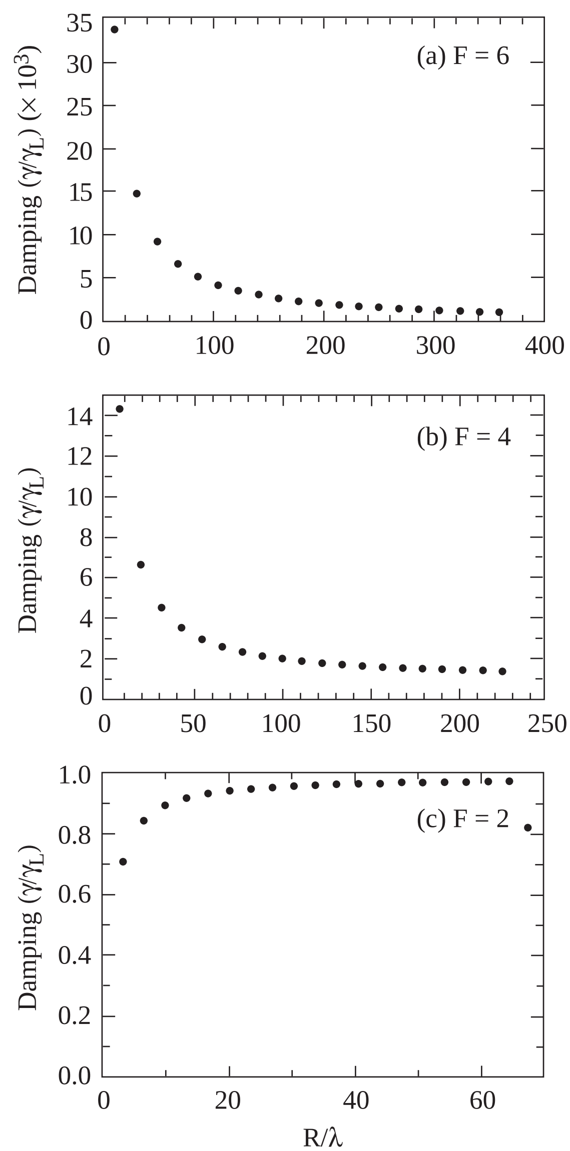

Figure 75.39
Same as Fig. 75.37, but for a cylinder of radius R. Here $F = \omega/kv_t$, where k is the radial wave number of the oscillation and the axial wave number is zero.

Conclusions

In summary, we have demonstrated a new, simplified approach to calculating transit-time damping. Our approach uses the time-reversal invariance of the Vlasov equation to avoid the necessity of calculating the wave–particle energy exchange to second order in the wave fields. We have illustrated the method by analyzing the damping of electrostatic oscillations in slab, cylindrical, and spherical geometries, both analytically and numerically. In general, our results seem to show that finite geometry effects tend to augment Landau damping when it would be small in an unbounded geometry, and reduce it when it would be large.

These results suggest a qualitative physical interpretation based on regarding the particles interacting with the electrostatic wave as falling into two classes: resonant and nonresonant. Resonant particles are those whose (unperturbed) motion keeps them in a constant phase relationship with the wave; depending on this phase they continuously either gain or lose energy from their interaction with the wave. As is well known, these are the particles responsible for Landau damping in infinite homogeneous plasmas. Nonresonant particles, on the other hand, see a varying wave phase as they propagate, and alternately gain and lose energy as this phase changes. In the case of an infinite geometry, these gains and losses cancel out over the infinite "transit time," and the nonresonant particles make no contribution to Landau damping. In the case of a finite system, the "resonant" particles can be regarded as those that do not get significantly out of phase with the wave while passing through the system; since their transit time decreases as the system becomes smaller, the number of particles that can be regarded as resonant increases as the confinement volume shrinks. It can be shown,[6] however, that the contribution of these nearly resonant particles to the damping goes as the fourth power of the time, so that the net contribution to the damping of the near-resonant particles diminishes as the confinement volume and the transit time become smaller. On the other hand, for a finite volume the energy gains and losses of the nonresonant particles no longer average to zero, and as the volume becomes smaller, the contribution of these nonresonant particles to the damping becomes larger. Thus, the damping in a finite system contains a smaller resonant component and a larger nonresonant component than in the corresponding infinite system. When the Landau damping is large in the infinite system (F small), the decrease in the resonant damping dominates the increase in the nonresonant damping, so that the damping in the finite system decreases from the Landau rate as the system size diminishes. When Landau damping is small (F large), the increase in nonresonant damping dominates, and the transit-time damping of the finite system is larger than the Landau damping of the corresponding infinite system. This picture is in qualitative agreement with the results we have obtained above for the slab, cylinder, and spherical geometries.

It should be noted that the essential advantage of the time-reversal invariance approach—the need to calculate the wave–particle energy transfer ΔE to only first order—is not dependent on the particular geometry of the system under consideration. For purposes of illustration, we have chosen simple geometries; in more complex geometries and inhomogeneous plasmas the phase-space integrals such as Eq. (13) will have to be carried out numerically, but the simplification in the calculation of ΔE will then be even more valuable. In Appendix B we show that the time-reversal invariance approach can be applied in quite general geometries, and verify that it gives results identical to the perturbed orbit approach.

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Appendix A: Large-Radius Limit of Collisionless Damping in Spherical Geometry

To evaluate the damping rate for large radii, we first investigate the nature of the function

$$S(kR,z) \equiv \frac{k}{R}\int_0^R db\,bG^2(kR,kb,z), \tag{A1}$$

from which Eqs. (14) and (15) contain the R dependence of the damping rate (here $z \equiv \omega/kv$ and the factor k is included for convenience to make the function dimensionless). From Eq. (10) we have, using $j_0(x) \equiv \sin(x)/x$ and defining $t = s/kR$ and $x = b/R$,

$$G(kR,x,z) = 2\int_0^{\sqrt{1-x^2}} \frac{\sin kR\sqrt{t^2+x^2}}{\sqrt{t^2+x^2}}\cos(zkRt)\,dt$$

$$= \mathrm{Im}\left\{\int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{t^2+x^2}} \left[e^{ikR\left(\sqrt{t^2+x^2}-zt\right)} + e^{ikR\left(\sqrt{t^2+x^2}+zt\right)}\right]dt\right\}$$

$$= \mathrm{Im}\left\{\int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{t^2+x^2}}\left[e^{ikR\psi_-(t)} + e^{ikR\psi_+(t)}\right]dt\right\}, \tag{A2}$$

where

$$\psi_{\pm}(t) \equiv \sqrt{t^2 + x^2} \pm zt\,.$$

We next use the method of stationary phase to determine the dominant behavior of $G(kR,x,z)$ as $kR \to \infty$. Using the Riemann–Lebesgue lemma, it is readily shown that the integral in Eq. (A2) vanishes as $1/kR$ as $kR \to \infty$ unless the functions

$$\psi'_{\pm}(t) = \frac{d}{dt}\left[\sqrt{t^2 + x^2} \pm zt\right] = \frac{t}{\sqrt{t^2 + x^2}} \pm z$$

vanish at some point in the t integration interval $\left[0, \sqrt{1 - x^2}\right]$, in which case the integral will vanish more slowly than $1/kR$ as $kR \to \infty$. Clearly $\psi'_{+}(t)$ cannot vanish in this interval, so the dominant behavior of G is given by

$$G(kR, x, z) \sim \mathrm{Im}\left\{\int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{t^2 + x^2}} e^{ikR\psi_{-}(t)} dt\right\} \quad \text{for} \quad x^2 + z^2 \le 1. \tag{A3}$$

The inequality in Eq. (A3) is the necessary and sufficient condition that $\psi'_{-}(t)$ vanish in $\left[0, \sqrt{1 - x^2}\right]$. When this inequality is not satisfied, G vanishes more rapidly as $kR \to \infty$ and hence may be neglected; thus, the dominant behavior of Eq. (A1) as $kR \to \infty$ is given by

$$S(kR, z) \sim kR\int_0^{\sqrt{1-z^2}} dx\, x\, G^2(kR, x, z). \tag{A4}$$

The dominant contribution to the integral in Eq. (A3) comes from the point in $\left[0, \sqrt{1 - x^2}\right]$ where $\psi'_{-}(t)$ vanishes, so we may extend the upper limit of the range of integration without changing the leading behavior of G:

$$G(kR, x, z) \sim \mathrm{Im}\left\{\int_0^1 \frac{1}{\sqrt{t^2 + x^2}} e^{ikR\left(\sqrt{t^2+x^2} - zt\right)} dt\right\} \quad \text{for} \quad x^2 + z^2 \le 1,$$

$$\sim 2\int_0^1 \frac{\sin\left(kR\sqrt{t^2 + x^2}\right)}{\sqrt{t^2 + x^2}} \cos(zkRt)\, dt.$$

Using $t = s/kR$ and $x = b/R$, this can be written

$$G(kR, kb, z) \sim 2\int_0^{kR} \frac{\sin\sqrt{k^2b^2 + s^2}}{\sqrt{k^2b^2 + s^2}} \cos(zs)\, ds$$

$$\sim 2\int_0^{\infty} \frac{\sin\sqrt{k^2b^2 + s^2}}{\sqrt{k^2b^2 + s^2}} \cos(zs)\, ds$$

$$= \pi J_0\left(kb\sqrt{1 - z^2}\right) \quad \text{for} \quad \left(\frac{\mathrm{b}}{\mathrm{R}}\right)^2 + z^2 \le 1.$$

Substituting in Eq. (A4), we obtain

$$S(kR, z) \sim \frac{1}{kR}\int_0^{kR\sqrt{1-z^2}} d(kb)(kb)\, G^2(kR, kb, z)$$

$$\sim \frac{\pi^2}{kR}\int_0^{kR\sqrt{1-z^2}} d(kb)(kb)\, J_0^2\left(kb\sqrt{1 - z^2}\right)$$

for $z \le 1$. Using

$$\int x\, J_0^2(\alpha x)\, dx = \frac{x^2}{2}\left[J_0^2(\alpha x) + J_1^2(\alpha x)\right],$$

this becomes

$$S(kR, z) \sim \frac{\pi^2 kR}{2}\left(1 - z^2\right)$$

$$\left\{J_0^2\left[kR\left(1 - z^2\right)\right] + J_1^2\left[kR\left(1 - z^2\right)\right]\right\}$$

$$\sim \pi \quad \text{as} \quad kR \to \infty \text{ for } z < 1,$$

where we have used the formula

$$\lim_{x\to\infty} x\left[J_0^2(x) + J_1^2(x)\right] = \frac{2}{\pi}.$$

For $z > 1$, since the condition $x^2 + z^2 \le 1$ cannot be satisfied, $S(kR,z)$ must vanish as $kR \to \infty$. Defining

$$T(z) \equiv \lim_{kR\to\infty} S(kR, z) = \begin{cases} \pi, & z < 1 \\ 0, & z > 1 \end{cases},$$

we see that $T(z)$ is a step function in z.

Thus, using Eqs. (14) and (15), the damping rate for large kR becomes

$$\lim_{kR\to\infty}\frac{\gamma}{\omega}=\lim_{kR\to\infty}\frac{P}{2\omega W}$$

$$=\frac{4\pi^2\omega e^2}{k^3}n_0\int_0^\infty dvv\left[-\frac{\partial f_0}{\partial E}\right]T\left(\frac{\omega}{kv}\right)$$

$$=\frac{4\pi^2\omega e^2}{k^3}n_0\pi\int_{\omega/k}^\infty dvv\left[-\frac{\partial f_0}{\partial E}\right]. \quad \text{(A5)}$$

Note from Eqs. (12) that f_0 here is the normalized three-dimensional distribution function, assumed isotropic:

$$4\pi\int_0^\infty v^2 f_0(v)\,dv=1.$$

Using

$$\frac{\partial f_0}{\partial E}=\frac{1}{mv}\frac{\partial f_0}{\partial v},$$

the integral in Eq. (A5) is readily evaluated to give

$$\lim_{kR\to\infty}\frac{\gamma}{\omega}=\frac{4\pi^3\omega e^2}{k^3}\,n_0 f_0\left(\frac{\omega}{k}\right). \quad \text{(A6)}$$

This result can be expressed in a more familiar form in terms of the one-dimensional velocity distribution g, defined by

$$g(u)\equiv 2\pi\int_0^\infty dvvf_0\left(\sqrt{u^2+v^2}\right)$$

$$=2\pi\int_u^\infty dvvf_0(v). \quad \text{(A7)}$$

Differentiating this expression gives

$$\frac{dg}{du}=-2\pi uf_0(u),\text{ or }\ f_0(u)=-\frac{1}{2\pi u}\frac{dg}{du}.$$

In terms of the one-dimensional distribution function, Eq. (A6) becomes

$$\lim_{kR\to\infty}\frac{\gamma}{\omega}=-\frac{2\pi^2e^2n_0}{k^2m}\left.\frac{dg(u)}{du}\right|_{u=\frac{\omega}{k}}$$

$$=-\frac{\pi\omega_p^2}{2k^2}\left.\frac{dg(u)}{du}\right|_{u=\frac{\omega}{k}},$$

which is just the Landau damping rate for plane waves of frequency ω and wave number k. This is to be expected since, as the radius of the sphere increases, an increasingly large fraction of the volume of the sphere contains waves that are locally planar, so that particles gain energy from them at the same rate as from a plane wave.

Appendix B: Equivalence of Perturbed Orbit and Time-Reversal Invariance Approaches to Transit-Time Damping

Transit-time damping of a confined electrostatic wave in a plasma arises from the transfer of energy from the wave to particles passing through the confinement region. In many cases of interest it may be assumed for purposes of calculating the damping that the wave properties (amplitude, frequency, etc.) are stationary in time. This means that background plasma properties such as the size and density of the confinement region are either constant or their variation is small during the wave period and the particle transit time. It also means that the wave energy lost to the damping is either replaced by another process, such as stimulated scattering, or again is small during the wave period and particle transit time.

Previous calculations of transit-time damping have taken a straightforward approach: the energy gained or lost by a particle transiting the confinement region is calculated, averaged over the phase of the wave, and integrated over the flux of particles weighted by the velocity distribution function. This approach can be represented in general by Fig. 75.40(a), and the power transferred from the wave to particles can be written

$$P=\int_0^\infty dv_x\,v_x\int d\mathbf{s}\,f_0(E)\left\langle\Delta E(E,0,\mathbf{s},\phi)\right\rangle_\phi$$

$$-\int_{-\infty}^0 dv_x\,v_x\int d\mathbf{s}\,f_0(E)\left\langle\Delta E(E,l,\mathbf{s},\phi)\right\rangle_\phi. \quad \text{(B1)}$$

Here the angle brackets denote averaging over the phase ϕ of the wave, and $\mathbf{s}$ represents the coordinates and velocities perpendicular to the arbitrarily chosen x axis:

$$d\mathbf{s}=dydzdv_ydv_z\,.$$

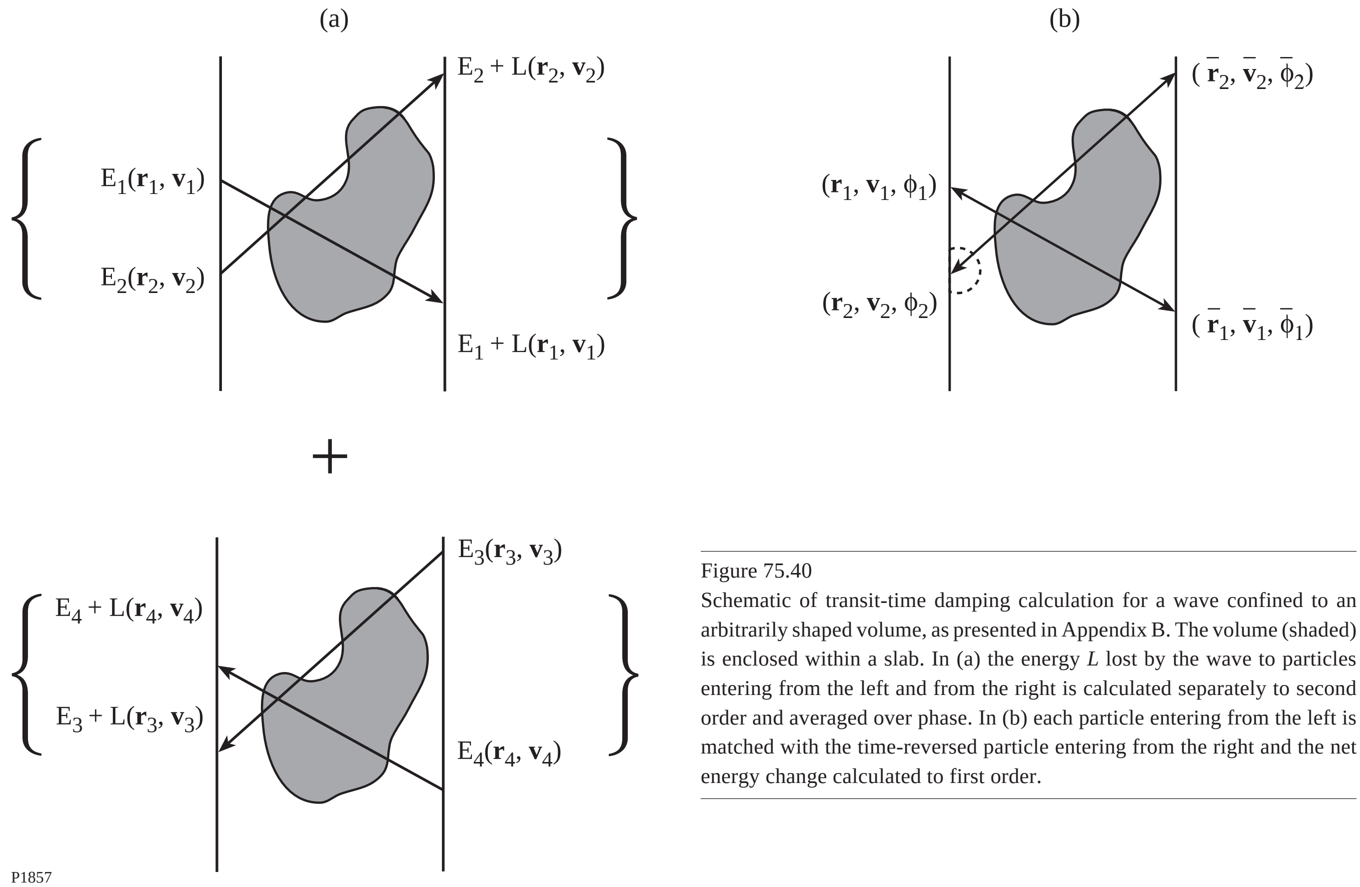

Figure 75.40
Schematic of transit-time damping calculation for a wave confined to an arbitrarily shaped volume, as presented in Appendix B. The volume (shaded) is enclosed within a slab. In (a) the energy L lost by the wave to particles entering from the left and from the right is calculated separately to second order and averaged over phase. In (b) each particle entering from the left is matched with the time-reversed particle entering from the right and the net energy change calculated to first order.

We include in our analysis all particles passing through a slab extending from $x = 0$ to $x = l$ and containing the confinement volume V. (Of course, only those particles following trajectories passing through V actually contribute to the damping, but describing only these trajectories is difficult for a volume of arbitrary shape. Including all trajectories passing through the slab greatly simplifies the representation of the particle flux in the general case and does not change the result since the additional trajectories do not contribute to the damping.) The functions $\Delta E\,(E,0,\mathbf{s},\phi)$ and $\Delta E\,(E,l,\mathbf{s},\phi)$ give the energy change for particles entering the slab at $x = 0$ and $x = l$, respectively, with energy E, phase ϕ, and other parameters $\mathbf{s}$. The distribution function f_0 is assumed uniform and isotropic and depends only on the energy $E = m\left(v_x^2 + v_y^2 + v_z^2\right)/2$.

The next step is to calculate the phase-averaged energy change:

$$L(E,0,\mathbf{s}) = \left\langle \Delta E(E,0,\mathbf{s},\phi)\right\rangle_\phi \text{ for } x = 0,\ v_x > 0;$$
$$L(E,l,\mathbf{s}) = \left\langle \Delta E(E,l,\mathbf{s},\phi)\right\rangle_\phi \text{ for } x = l,\ v_x < 0.$$

The energy ΔE gained or lost by a specific particle is first order in the field amplitude, but the gains and losses cancel to first order after phase averaging, so that the loss functions L are second order in the field. Evaluation of the loss functions thus requires that the energy changes ΔE also be calculated to second order, which in turn means that the perturbed trajectories must be determined and integrated over. This can lead to complicated calculations in general. Details of the calculation of the loss functions and the resulting damping rates are given for some simple cases in Robinson.[9]

Our purpose here is to show that the integrations in Eq. (B1) can be rearranged so that ΔE need only be calculated to first order, which can be accomplished by integration over the unperturbed orbits.

First we take the phase average outside the integrations and write it explicitly as an integral over ϕ:

$$P = \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_0^{\infty} dv_x\, v_x \int d\mathbf{s}\, f_0(E)\,\Delta E(E,0,\mathbf{s},\phi)$$
$$- \frac{1}{2\pi}\int_0^{2\pi} d\bar{\phi} \int_{-\infty}^{0} d\bar{v}_x\, \bar{v}_x \int d\,\bar{\mathbf{s}}\, f_0\left(\bar{E}\right)\Delta\bar{E}\left(\bar{E},l,\bar{\mathbf{s}},\bar{\phi}\right), \qquad \text{(B2)}$$

where we have also denoted the integration parameters for particles entering the slab from the right at $x = l$ by an overbar (this amounts only to a change of dummy variable at this point and has no physical significance). We could, however, just as well calculate the second integral in Eq. (B2) by integrating over the parameters with which these particles leave the slab at $x = 0$. Since we are dealing with a collisionless plasma, we can invoke Liouville's theorem to say that an element of phase-space volume is invariant on passing through the slab:

$$dxdydzdv_x dv_y dv_z = d\bar{x}\, d\bar{y}\, d\bar{z}\, d\bar{v}_x\, d\bar{v}_y\, d\bar{v}_z\,. \qquad \text{(B3)}$$

Using $d\phi = \omega\, d\,t$, $d\bar{\phi} = \omega\, d\,\bar{t}$, $dx = v_x dt$, and $d\bar{x} = \bar{v}_x d\bar{t}$, where ω is the wave frequency, Eq. (B3) becomes

$$v_x dydzdv_x dv_y dv_z d\phi = \bar{v}_x\, d\bar{y}\, d\bar{z}\, d\bar{v}_x\, d\bar{v}_y\, d\bar{v}_z\, d\bar{\phi}$$

or

$$v_x dv_x d\,\mathbf{s}\, d\phi = \bar{v}_x d\,\bar{v}_x d\,\bar{\mathbf{s}}\, d\bar{\phi}\,. \qquad \text{(B4)}$$

Thus, the transformation from the integration parameters at $x = l$ to those at $x = 0$ has unit Jacobian, and we can write Eq. (B2) as

$$P = \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_0^{\infty} dv_x\, v_x \int d\mathbf{s}\, f_0(E)\,\Delta E(E,0,\mathbf{s},\phi)$$
$$-\frac{1}{2\pi}\int_0^{2\pi} d\phi \int_{-\infty}^{0} dv_x\, v_x \int d\mathbf{s}\, f_0(\bar{E})\,\Delta\bar{E}(\bar{E},l,\bar{\mathbf{s}},\bar{\phi}), \qquad \text{(B5)}$$

where $\bar{E}$ is now a function of the $x = 0$ parameters:

$$\bar{E} = \frac{1}{2}m\left(v_x^2 + v_y^2 + v_z^2\right) + \Delta\, E(E,0,\mathbf{s},\phi)$$
$$= E + \Delta\, E(E,0,\mathbf{s},\phi). \qquad \text{(B6)}$$

Also, from the definitions of ΔE and $\Delta\bar{E}$, we have

$$\Delta\,\bar{E}\left(\bar{E},l,\bar{\mathbf{s}},\bar{\phi}\right) = E - \bar{E} = -\Delta\, E(E,0,\mathbf{s},\phi). \qquad \text{(B7)}$$

Substituting Eqs. (B6) and (B7) in Eq. (B5), we get

$$P = \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_0^{\infty} dv_x\, v_x \int d\mathbf{s}\, f_0(E)\,\Delta E(E,0,\mathbf{s},\phi)$$
$$+\frac{1}{2\pi}\int_0^{2\pi} d\phi \int_{-\infty}^{0} dv_x\, v_x \int d\mathbf{s}\, f_0(E+\Delta E)\,\Delta E(E,0,\mathbf{s},\phi). \qquad \text{(B8)}$$

Since the process is assumed to be stationary, Eq. (B8) must be invariant under time reversal. The only effect of the time-reversal operator on Eq. (B8) is to change the sign of v_x (strictly speaking, it also changes the phase by a constant, but since we are integrating over all ϕ, this is irrelevant). The time-reversed form of Eq. (B8) is thus

$$P = -\frac{1}{2\pi}\int_0^{2\pi} d\phi \int_{-\infty}^{0} dv_x\, v_x \int d\mathbf{s}\, f_0(E)\,\Delta E(E,0,\mathbf{s},\phi)$$

$$-\frac{1}{2\pi}\int_0^{2\pi} d\phi \int_0^{\infty} dv_x\, v_x \int d\mathbf{s}\, f_0(E+\Delta\, E)\,\Delta E(E,0,\mathbf{s},\phi). \qquad \text{(B9)}$$

Adding Eqs. (B8) and (B9) and dividing by 2 gives

$$P = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dv_x\, |v_x|$$

$$\int d\mathbf{s}\left[f_0(E) - f_0(E+\Delta\, E)\right]\Delta E(E,0,\mathbf{s},\phi)$$

$$= -\frac{1}{2}\int_{-\infty}^{\infty} dv_x\, |v_x| \int d\mathbf{s} \left\langle \Delta E^2(E,0,\mathbf{s},\phi)\right\rangle_\phi \frac{df_0}{dE}. \qquad \text{(B10)}$$

Note that although this expression is second order in the field, as it should be, it achieves second order only through the squaring of ΔE, so that ΔE itself need only be calculated to first order.

Equation (B10) is a surface integral, i.e., the values of v_x and $\mathbf{s}$ in the integral are evaluated on the $x = 0$ surface of the slab. It is useful to rewrite Eq. (B10) in a form involving a volume integral rather than a flux. The integration in Eq. (B10) is shown schematically in Fig. 75.40(b). Since we are calculating ΔE to first order, we can represent the particle trajectories by their unperturbed orbits. Consider the six-dimensional "flux tube" traced out by a phase-space volume element crossing the slab along an unperturbed orbit (which need not be a straight line). The rate at which phase-space volume enters the tube is $v_x d\mathbf{s}$, and since in a collisionless process phase-space volume is conserved, the volume of the flux tube is given by

$$\Delta V = t_0(E,\mathbf{s})\, v_x d\mathbf{s}, \qquad \text{(B11)}$$

where $t_0(E,\mathbf{s})$ is the time taken for a particle following the orbit to cross the slab. Since phase-space volume moves as an incompressible fluid, flux tubes cannot intersect, and a set of these flux tubes whose collective cross section comprises the $x = 0$ plane will exactly fill the phase-space volume within the slab. Furthermore, the (unperturbed) flux $v_x d\mathbf{s}$ through the tube is a constant, so we may deform the slab boundary as shown by the dotted contour in Fig. 75.40(b) without affecting the validity of Eq. (B11); the volume of the tube and the time taken to pass along it are reduced in the same proportion. As long as the deformed boundary is outside the volume V in which the potential is nonvanishing, ΔE is also unaffected, so we may deform the original slab boundary to conform to the boundary of V and use Eq. (B11) to convert Eq. (B10) to an integral over the phase space within V:

$$P = -\frac{1}{2}\int_V d^3\mathbf{r}\int d^3\mathbf{v}\,\frac{\left\langle \Delta E^2(\mathbf{r},\mathbf{v},\phi)\right\rangle_\phi}{t_0(\mathbf{r},\mathbf{v})}\,\frac{df_0}{dE}$$

$$= \frac{1}{2}\int d^3\mathbf{r}\int d^3\mathbf{v}\Delta P, \qquad \text{(B12)}$$

where ΔP is the expression for the energy loss for a volume of phase space we wrote down immediately on the basis of time-reversal invariance in Eq. (1) at the beginning of this article. We have derived Eq. (B12) from Eq. (B1) here to demonstrate the equivalence of our approach to earlier formulations of transit-time damping, which are also based on Eq. (B1).

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