Accurate Formulas for the Landau Damping Rates of Electrostatic Waves

Laser–plasma instabilities[1] are important in the field of inertial confinement fusion[2] because they scatter laser light away from the target, which reduces the laser energy available to drive the compression of the nuclear fuel, or generate energetic electrons that preheat the fuel, which makes the fuel harder to compress. In stimulated Raman scattering an incident, or pump, light wave (0) decays into a frequency-downshifted, or Stokes, light wave (1) and an electron-plasma wave (2). In stimulated Brillouin scattering a pump light wave decays into a Stokes light wave and an ion-acoustic wave (2). The initial evolution of both instabilities is governed by the linearized equations[3]

$$\left(\partial_t + v_1 \partial_z\right) A_1 = \gamma_0 A_2 , \tag{1}$$

$$\left(\partial_t + \nu_2\right) A_2 = \gamma_0 A_1 , \tag{2}$$

where A_1 and v_1 are the amplitude and group speed of the Stokes wave, respectively, A_2 and ν_2 are the amplitude and damping rate of the plasma wave (electron-plasma or ion-acoustic), respectively, and the coupling constant γ_0 is proportional to the amplitude of the pump wave. The convective amplification of an existing Stokes wave and the generation of a Stokes wave by plasma fluctuations are both characterized by the gain exponent[3]

$$g = \gamma_0^2 l / v_1 \nu_2 , \tag{3}$$

where l is the plasma length. Because the aforementioned parametric instabilities are important only when $g >> 1$, a small error in the damping rate of the plasma wave can produce a large error in the predicted amplitude of the Stokes wave. For this reason, it is important to determine accurately the Landau contribution to the damping rates of plasma waves.[4]

The properties of electrostatic plasma waves are determined by the dielectric function

$$\varepsilon(\omega, k) = 1 + \chi_e(\omega, k) + \chi_i(\omega, k), \tag{4}$$

where χ_e and χ_i denote the electron and ion susceptibilities, respectively. For each species s

$$\chi_s(\omega, k) = -\frac{\omega_s^2}{2 v_s^2 k^2} Z'\left(\frac{\omega}{\sqrt{2} v_s k}\right), \tag{5}$$

where ω_s is the plasma frequency, v_s is the thermal speed, and Z is the plasma dispersion function.[5] The electrostatic dispersion equation is simply

$$\varepsilon(\omega, k) = 0 . \tag{6}$$

The solution of this dispersion equation has two branches: the high-frequency (electron-plasma) branch and the low-frequency (ion-acoustic) branch, both of which are studied in this article. In both cases our approximate analytical solution of the dispersion equation is compared to the numerical solution. Our analytical solutions are more accurate than the standard analytical solutions found in textbooks.[6–8]

Electron-Plasma Waves

The electron Debye length $\lambda_e = v_e / \omega_e$. For the case in which $k\lambda_e << 1$, Krall and Trivelpiece,[6] Ichimaru,[7] and Chen[8] all assert that

$$\omega_r \approx \omega_e \left[1 + 3\left(k\lambda_e\right)^2\right]^{1/2} , \tag{7}$$

$$\omega_i \approx -\left(\frac{\pi}{8}\right)^{1/2} \frac{\omega_e}{\left(k\lambda_e\right)^3} \exp\left[-\frac{1}{2\left(k\lambda_e\right)^2} - \frac{3}{2}\right]. \tag{8}$$

To gauge the accuracy of these formulas, we considered a numerical example. When $(k\lambda_e)^2 = 0.1$, formula (7) predicts that $\omega_r/\omega_e \approx 1.140$. The correct value of this frequency ratio, obtained by solving Eq. (4) numerically, with the ion term omitted, is 1.179. Formula (8) predicts that $\omega_i/\omega_e \approx 0.02979$, whereas the correct value is 0.01845. Although the predicted frequency is in error by only 3.3%, the predicted damping rate is in error by 61%. Clearly there is room for improvement.

In the aforementioned parameter regime $\omega \approx \omega_e$ and $\omega/v_e k \approx 1/(k\lambda_e) >> 1$. The electron-plasma dispersion function has the asymptotic expansion[5]

$$Z(\varsigma) \sim i\sigma\pi^{1/2}\exp\left(-\varsigma^2\right) - \sum_{n=0}^{\infty}\Gamma(n+1/2)\Big/\left[\Gamma(1/2)\varsigma^{2n+1}\right], \quad (9)$$

where

$$\sigma = \begin{cases} 0, & \text{if } \varsigma_i > 1/|\varsigma_r|, \\ 1, & \text{if } |\varsigma_i| < 1/|\varsigma_r|, \\ 2, & \text{if } \varsigma_i < -1/|\varsigma_r|, \end{cases} \quad (10)$$

$\Gamma(n+1/2) = (n-1/2)\Gamma(n-1/2)$ and $\Gamma(1/2) = \pi^{1/2}$. It is convenient to introduce the dimensionless parameters $K = k\lambda_e$ and $\Omega = \omega/\omega_e$. If one neglects the ion term in Eq. (4), the electron-plasma dispersion equation can be written as

$$D_r(\Omega) + iD_i(\Omega) = 0, \quad (11)$$

where

$$D_r(\Omega) = 1 - \sum_{n=1}^{\infty}(2n-1)!!\,K^{2n-2}/\Omega^{2n}, \quad (12)$$

$(2n-1)!! = (2n-1)(2n-3)\ldots(3)(1)$, and

$$D_i(\Omega) = \left(\frac{\pi}{2}\right)^{1/2}\frac{\Omega}{K^3}\exp\left(-\frac{\Omega^2}{2K^2}\right). \quad (13)$$

Because the exponent in Eq. (13) is proportional to $1/K^2$, $|D_i/D_r|$ and $|\Omega_i|/\Omega_r$ are exponentially small when $K^2 << 1$.

If one assumes that $|\Omega_i|/\Omega_r$ is less than any power of K required for an accurate solution of Eq. (11), then Ω_r is determined by the equation

$$D_r(\Omega_r) = 0 \quad (14)$$

and Ω_i is given by the formula

$$\Omega_i \approx -\frac{D_i}{\partial D_r/\partial\Omega}\bigg|_{\Omega_r}. \quad (15)$$

By using Eq. (12) to evaluate the derivative in Eq. (13), one finds that

$$\Omega_i \approx -\left(\frac{\pi}{8}\right)^{1/2}\frac{C(\Omega_r)}{K^3}\exp\left(-\frac{\Omega_r^2}{2K^2}\right), \quad (16)$$

where the coefficient function

$$C(\Omega) = \Omega^4\bigg/\left[\sum_{n=1}^{\infty}n(2n-1)!!\,K^{2n-2}/\Omega^{2n-2}\right]. \quad (17)$$

It is clear from Eqs. (12) and (17) that the dispersion equation (14) is an equation for Ω_r^2 that involves the small parameter K^2, and formula (16) depends on Ω_r^2. The efficient way to proceed is to solve Eq. (14) and evaluate formula (16) perturbatively, by expanding Ω_r^2 and D_r in powers of K^2. We chose to expand Ω_r and D_r in powers of K^2 to facilitate the analysis in the **Ion-Acoustic Waves** section. Specifically, we made the third-order expansions

$$\Omega \approx 1 + K^2\Omega_1 + K^4\Omega_2 + K^6\Omega_3, \quad (18)$$

$$D(\Omega) \approx D_0(\Omega) + K^2D_1(\Omega) + K^4D_2(\Omega) + K^6D_3(\Omega), \quad (19)$$

where $D_0(\Omega) = 1 - 1/\Omega^2$, $D_1(\Omega) = -3/\Omega^4$, $D_2(\Omega) = -15/\Omega^6$, and $D_3(\Omega) = -105/\Omega^8$, and

$$D_n(\Omega) \approx D_n(1) + D_n'(1)(\Omega-1) + D_n''(\Omega-1)^2/2 + D_n'''(1)(\Omega-1)^3/6. \quad (20)$$

We substituted these expansions in Eq. (14) and collected terms of like order.

The zeroth-order equation is satisfied identically. The first-order equation is

$$D_0'\Omega_1 + D_1 = 0, \tag{21}$$

from which it follows that

$$\Omega_1 = 3/2. \tag{22}$$

The second-order equation is

$$D_0'\Omega_2 + D_0''\Omega_1^2/2 + D_1'\Omega_1 + D_2 = 0, \tag{23}$$

from which it follows that

$$\Omega_2 = 15/8. \tag{24}$$

The third-order equation is

$$\begin{aligned} D_0'\Omega_3 + D_0''\Omega_2\Omega_1 + D_0'''\Omega_1^3/6 + D_1'\Omega_2 \\ + D_1''\Omega_1^2/2 + D_2'\Omega_1 + D_3 = 0, \end{aligned} \tag{25}$$

from which it follows that

$$\Omega_3 = 147/16. \tag{26}$$

By combining Eqs. (22), (24), and (26), one finds that

$$\Omega_r \approx 1 + 3K^2/2 + 15K^4/8 + 147K^6/16, \tag{27}$$

from which it follows that

$$\Omega_r^2 \approx 1 + 3K^2 + 6K^4 + 24K^6. \tag{28}$$

Since the exponent in Eq. (16) is proportional to $1/K^2$, the third-order formula for Ω_r^2 determines the exponential term correct to second order. Consequently, one need only determine C correct to second order.[9] The result is

$$C \approx 1 - 6K^4. \tag{29}$$

It follows from Eqs. (16), (28), and (29) that

$$\begin{aligned} \Omega_i \approx -\left(\frac{\pi}{8}\right)^{1/2}\left(\frac{1}{K^3} - 6K\right) \\ \times \exp\left(-\frac{1}{2K^2} - \frac{3}{2} - 3K^2 - 12K^4\right). \end{aligned} \tag{30}$$

We refer to formulas (27) and (30) as the third-order formulas, even though the latter formula is only accurate to second order. In a similar way, one could refer to the textbook formulas as the first-order formulas. Notice, however, that the textbook formula $\Omega_r \approx (1+3K)^{1/2}$ is less accurate than the true first-order formula $\Omega_r \approx 1 + 3K/2$.

The approximate analytical solutions of the electron-plasma dispersion equation are compared to the numerical solution in Fig. 74.51. The dashed lines represent the textbook solution, the solid lines represent the third-order solution, and the dotted lines represent the numerical solution. For $K^2 = 0.1$ the third-order formulas predict that $\Omega_r \approx 1.178$ and $\Omega_i \approx 0.01840$. These values of Ω_r and Ω_i differ from the correct values by 0.085% and 0.27%, respectively. For the displayed range of K^2 the maximal error associated with the third-order formula for Ω_r is 0.57% and the maximal error associated with the third-order formula for Ω_i is 14%. The third-order formulas are more accurate than the textbook formulas, even though the assumption on which they are based, that $|\Omega_i|/\Omega_r << K^6$, is only valid for $K^2 < 0.04$. Neither pair of formulas is accurate when K^2 is significantly larger than 0.1.

Ion-Acoustic Waves

The electron contribution to the ion-acoustic speed $c_e = \left(ZT_e/m_i\right)^{1/2}$, where Z is the ionization number. For the case in which $k\lambda_e << 1$ and $T_i/ZT_e << 1$, we define the baseline formulas

$$\omega_r \approx c_e k\left(1 + 3T_i/ZT_e\right)^{1/2}, \tag{31}$$

$$\frac{\omega_i}{c_e k} \approx -\left(\frac{\pi}{8}\right)^{1/2}\left[\left(\frac{Zm_e}{m_i}\right)^{1/2} + \left(\frac{ZT_e}{T_i}\right)^{3/2}\exp\left(-\frac{ZT_e}{2T_i} - \frac{3}{2}\right)\right]. \tag{32}$$

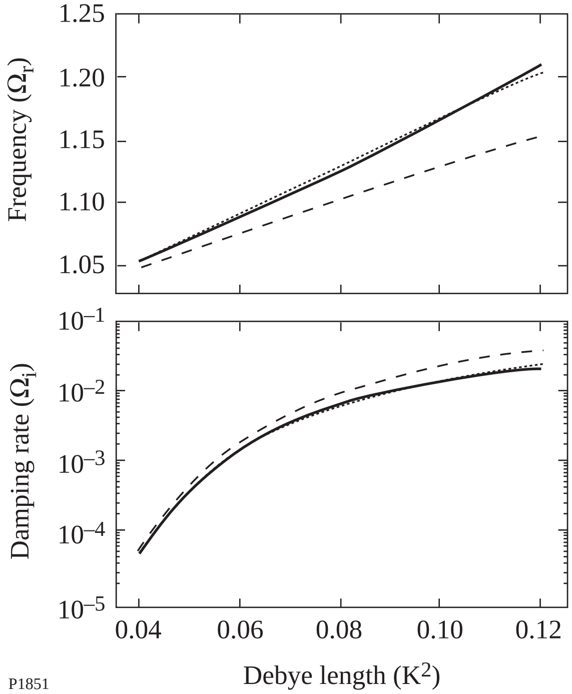

Figure 74.51
(a) Normalized frequency (ω_r/ω_e) and (b) damping rate (ω_i/ω_e) of an electron-plasma wave plotted as functions of the square of the normalized Debye length ($k\lambda_e$). The dashed lines represent the textbook formulas (7) and (8), the solid lines represent the third-order formulas (27) and (30), and the dotted lines denote numerical solutions of the electron-plasma dispersion equation.

Krall and Trivelpiece[6] omit the ion-temperature contribution to the frequency (31) and the associated factor of exp(–3/2) in the ion contribution to the damping rate (32). Ichimaru[7] and Chen[8] retain these ion-temperature contributions. They agree on formula (31) for the frequency but differ on the formula for the damping rate. Ichimaru multiplies formula (32) by a factor of $\left(1+3T_i/ZT_e\right)^{1/2}$, whereas Chen, who considers only the ion contribution to the damping rate, multiplies the ion term in formula (32) by a factor of $1+3T_i/ZT_e$. In a recent paper[10] we showed empirically that Ichimaru's formula for the damping rate is the better of the two. To gauge the accuracy of Ichimaru's formulas, we considered a numerical example: When $T_i/ZT_e = 0.1$, formula (31) predicts that $\omega_r/c_ek \approx 1.140$. The correct value of the frequency ratio, obtained by solving Eq. (4) numerically with $k\lambda_e = 0.001$, is 1.181. Formula (32), multiplied by $\left(1+3T_i/ZT_e\right)^{1/2}$, predicts that $\omega_i/c_ek \approx 0.05064$, whereas the correct value is 0.03219. Although the predicted frequency is only in error by 3.4%, the predicted damping rate is in error by 57%. [For comparison, the damping rate predicted by formula (32) is in error by 38%.] Clearly there is room for improvement in the accuracy of the formula for the damping rate and the self-consistency of the method by which it is derived.

In the aforementioned parameter regime $\omega \approx c_ek$, $\omega/v_ek \approx \left(Zm_e/m_i\right)^{1/2} << 1$, and $\omega/v_ik \approx \left(ZT_e/T_i\right)^{1/2} >> 1$. One can use the expansion[5]

$$Z(\varsigma) = i\pi^{1/2}\exp\left(-\varsigma^2\right) - \varsigma\sum_{n=0}^{\infty}\left(-\varsigma^2\right)^n\Gamma(1/2)/\Gamma(n+3/2) \quad (33)$$

for the electron-plasma dispersion function and expansion (9) for the ion-plasma dispersion function. It is convenient to introduce the dimensionless parameters $T = T_i/ZT_e$ and $\Omega = \omega/c_ek$.

If one makes the assumption that $Z'(\varsigma_e) \approx -2$, which omits the electron contribution to the ion-acoustic damping rate, the ion-acoustic dispersion relation can be written in the form of Eq. (11), where

$$D_r(\Omega) = 1 + K^2 - \sum_{n=1}^{\infty}(2n-1)!!\,T^{n-1}/\Omega^{2n}, \quad (34)$$

$$D_i(\Omega) = \left(\frac{\pi}{2}\right)^{1/2}\frac{\Omega}{T^{3/2}}\exp\left(-\frac{\Omega^2}{2T}\right). \quad (35)$$

Since the dispersion functions can be rewritten as

$$\frac{D_r(\Omega)}{\left(1+K^2\right)} = 1 - \sum_{n=1}^{\infty}\frac{(2n-1)!!\left[T\left(1+K^2\right)\right]^{n-1}}{\left[\Omega^2\left(1+K^2\right)\right]^n}, \quad (36)$$

$$\frac{D_i(\Omega)}{\left(1+K^2\right)} = \left(\frac{\pi}{2}\right)^{1/2}\frac{\left[\Omega^2\left(1+K^2\right)\right]^{1/2}}{\left[T\left(1+K^2\right)\right]^{3/2}}\exp\left[-\frac{\Omega^2\left(1+K^2\right)}{2T\left(1+K^2\right)}\right], \quad (37)$$

Ω_r and Ω_i satisfy the equation

$$\Omega[K,T] = \Omega\left[0, T\left(1+K^2\right)\right]\Big/\left(1+K^2\right)^{1/2}. \quad (38)$$

Thus, one need only solve the ion-acoustic dispersion equation for the case in which $K^2 = 0$. In this case Eq. (34) has the same form as Eq. (12), with K^2 replaced by T. It follows from this observation, and Eqs. (27) and (30), that the third-order solution is

$$\Omega_r \approx 1 + 3T/2 + 15T^2/8 + 147T^3/16, \tag{39}$$

$$\Omega_i \approx -\left(\frac{\pi}{8}\right)^{1/2}\left(\frac{1}{T^{3/2}} - 6T^{1/2}\right) \times \exp\left(-\frac{1}{2T} - \frac{3}{2} - 3T - 12T^2\right). \tag{40}$$

Equations (38)–(40) apply to all values of K^2 that satisfy the inequality $T(1 + K^2) << 1$.

If one makes the approximation $Z'(\varsigma_e) \approx -2 - i\pi^{1/2}\varsigma_e$, which retains the electron contribution to the ion-acoustic damping rate, one must add to Eq. (35) the term

$$D_i(\Omega) = i\left(\frac{\pi}{2}\right)^{1/2}\Omega M^{1/2}, \tag{41}$$

where $M = Zm_e/m_i$ and Z is the ionization number. Since

$$\frac{D_i(\Omega)}{1+K^2} = i\left(\frac{\pi}{2}\right)^{1/2}\frac{\left[\Omega^2\left(1+K^2\right)\right]^{1/2}M^{1/2}}{\left(1+K^2\right)^{3/2}}, \tag{42}$$

Ω_r and Ω_i satisfy the equation

$$\Omega[K,M,T] = \Omega\left[0, M\Big/\left(1+K^2\right)^3, T\left(1+K^2\right)\right]\Big/\left(1+K^2\right)^{1/2}. \tag{43}$$

Thus, one need only solve the ion-acoustic dispersion equation for the case in which $K^2 = 0$.

Unlike the ion contribution to D_i, the electron contribution is not exponentially small when $T << 1$, so one cannot evaluate formula (15) correct to an arbitrary power of T. This formula suggests, however, that the electron contribution to Ω_i is of order 0.01. It follows from Eq. (40) and Fig. 74.51(b) that the ion contribution to Ω_i is much smaller than the electron contribution for $T \le 0.06$ and is comparable to the electron contribution for $0.08 \le T \le 0.12$. In the latter range, both contributions to Ω_i are of order T^2. To make a perturbation expansion based on this ordering, we defined the damping parameters

$$\Gamma = \left(\frac{\pi}{8}\right)^{1/2}\frac{M^{1/2}}{T^2}, \tag{44}$$

$$\Delta = \left(\frac{\pi}{8}\right)^{1/2}\frac{1}{T^{7/2}}\exp\left(-\frac{1}{2T} - \frac{3}{2}\right) \tag{45}$$

and made the approximation

$$\exp\left(-\frac{\Omega^2}{2T}\right) \approx \exp\left(-\frac{1}{2T} - \frac{3}{2}\right)\left[1 - T\left(\Omega_2 + \frac{\Omega_1^2}{2}\right)\right], \tag{46}$$

which allowed us to write the real dispersion function as

$$D_r(\Omega) \approx D_{0r} + TD_{1r} + T^2 D_{2r} + T^3 D_{3r}, \tag{47}$$

where D_{0r}–D_{2r} were defined after Eq. (19) and

$$D_{3r} = -105 - 2\Delta\Omega_{2i}, \tag{48}$$

and the imaginary dispersion function as

$$D_i(\Omega) \approx T^2 D_{2i} + T^3 D_{3i}, \tag{49}$$

where

$$D_{2i} = 2(\Gamma + \Delta), \tag{50}$$

$$D_{3i} = 2\left[\Omega_{1r}(\Gamma + \Delta) - \Delta\left(\Omega_{2r} + \Omega_{1r}^2/2\right)\right]. \tag{51}$$

We then proceeded as described in the **Electron-Plasma Waves** section.

The zeroth-order and first-order equations are identical to the corresponding equations of the previous section, so $\Omega_{1r} =$

3/2 and $\Omega_{1i} = 0$ as we assumed in Eq. (46). The second-order equation is

$$\begin{aligned} & D'_{0r}\left(\Omega_{2r} + i\Omega_{2i}\right) \\ & + D''_{0r}\Omega_{1r}^2/2 + D'_{1r}\Omega_{1r} + D_{2r} + iD_{2i} = 0, \end{aligned} \tag{52}$$

from which it follows that

$$\Omega_{2r} = 15/8, \tag{53}$$

$$\Omega_{2i} = -(\Gamma + \Delta). \tag{54}$$

Formula (54) is equivalent to the base-line formula (32). The third-order equation is

$$\begin{aligned} & D'_{0r}\left(\Omega_{3r} + i\Omega_{3i}\right) \\ & + D''_{0r}\left(\Omega_{2r} + i\Omega_{2i}\right)\Omega_{1r} \\ & + D'''_{0r}\Omega_{1r}^3/6 \\ & + D'_{1r}\left(\Omega_{2r} + i\Omega_{2i}\right) \\ & + D''_{1r}\Omega_{1r}^2/2 + D'_{2r}\Omega_{1r} + D_{3r} + iD_{3i} = 0, \end{aligned} \tag{55}$$

from which it follows that

$$\Omega_{3r} = 147/16 + \Delta(\Gamma + \Delta), \tag{56}$$

$$\Omega_{3i} = \Delta\left(\Omega_{2r} + \Omega_{1r}^2/2\right). \tag{57}$$

By combining Eqs. (53) and (56), one finds that

$$\Omega_r \approx 1 + 3T/2 + 15T^2/8 + \left[147/16 + \Delta(\Gamma + \Delta)\right]T^3. \tag{58}$$

By combining Eqs. (54) and (57), one finds that

$$\Omega_i = -T^2\left\{\Gamma + \Delta\left[1 - T\left(\Omega_{2r} + \Omega_{1r}^2/2\right)\right]\right\}. \tag{59}$$

It is clear from Eq. (46) that the Δ terms represent the exponential $\exp\left(-\Omega_r^2/2T\right)$, with the exponent evaluated correct to first order; thus, one can rewrite Eq. (59) as

$$\Omega_i \approx -\left(\frac{\pi}{8}\right)^{1/2}\left[M^{1/2} + \frac{1}{T^{3/2}}\exp\left(-\frac{1}{2T} - \frac{3}{2} - 3T\right)\right]. \tag{60}$$

Notice that the algebraic factors of Ichimaru and Chen are both absent. We refer to formulas (58) and (60) as the third-order formulas, even though the exponent in the latter formula is only accurate to first order.

The approximate analytical solutions of the ion-acoustic dispersion equation are compared to the numerical solution in Fig. 74.52. The dashed lines represent Ichimaru's solution, the solid lines represent the third-order solution, and the dotted lines represent the numerical solution. For $T = 0.1$ formula (58) predicts that $\Omega_r \approx 1.191$, which differs from the correct value of Ω_r by 0.85%. It is clear from Figs. 74.51(a) and 74.52(a) that the additional third-order term improves the accuracy of the formula in the range $T \leq 0.09$ but decreases the accuracy in the

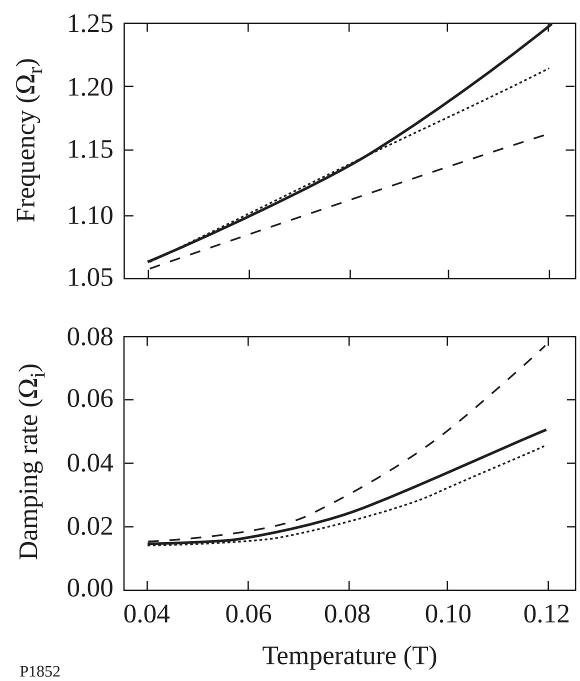

Figure 74.52
(a) Normalized frequency $\left(\omega_r/c_e k\right)$ and (b) damping rate $\left(\omega_i/c_e k\right)$ of an ion-acoustic wave plotted as functions of the temperature ratio $\left(T_i/ZT_e\right)$. The dashed lines represent Ichimaru's formulas (31) and (32) multiplied by $\left(1 + 3T_i/ZT_e\right)^{1/2}$, the solid lines represent the third-order formulas (58) and (60), and the dotted lines denote numerical solutions of the ion-acoustic dispersion equation.

range $T > 0.09$. Formula (60) predicts that $\Omega_i \approx 0.03670$, which differs from the correct value of Ω_i by 14%. For the displayed range of T the maximal error associated with the third-order formula for Ω_r is 2.7% and the maximal error associated with the third-order formula for Ω_i is 14%. The third-order formulas are more accurate than Ichimaru's formulas. Neither pair of formulas is accurate when T is significantly larger than 0.1.

Summary

We used systematic perturbation methods to derive formulas for the Landau damping rates of electron-plasma waves [Eq. (30)] and ion-acoustic waves [Eq. (60)]. The predictions of these formulas were compared to the predictions of the textbook formulas[6–8] and numerical solutions of the electrostatic dispersion equation. When $(k\lambda_e)2 \le 0.1$ (for electron-plasma waves) and $T_i/ZT_e \le 0.1$ (for ion-acoustic waves), our formulas are more accurate than the textbook formulas. When $(k\lambda_e)^2 > 0.1$ and $T_i/ZT_e > 0.1$, no pair of formulas is accurate and the electrostatic dispersion equation must be solved numerically.

ACKNOWLEDGMENT

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