Dephasing Time of an Electron Accelerated by a Laser Pulse

In a recent paper[1] we described the motion of electrons in the electromagnetic field of a circularly polarized laser pulse propagating through a plasma. Electrons that are in front of the pulse initially can be accelerated to high energies and extracted easily. Although this direct acceleration scheme is less than ideal because the pulse can generate a parasitic wake, its simplicity is noteworthy. The wake fields produced by short pulses have been observed recently,[2,3] and future experiments will study the interaction of electron bunches with these wake fields. One would only need to change the timing of an electron bunch in these experiments to test the scientific feasibility of direct acceleration. In this article we study the dephasing time of an electron accelerated by a pulse of infinite width to determine the propagation time and plasma length required to observe direct acceleration.

In the following sections, the trajectory of a charged particle is determined analytically for a representative pulse profile; the dephasing time of an accelerated particle is determined; its dependence on the speed, length, and intensity of the pulse along with the injection energy of the particle is studied in detail; and finally, the main results are summarized.

Particle Motion in a Planar Field

The motion of a particle, of charge q and mass m, in an electromagnetic field is governed by the equation[4]

$$d_\tau\left(u_\mu + a_\mu\right) = u^\nu \partial_\mu a_\nu\,, \tag{1}$$

where u^μ is the four-velocity of the particle divided by c, τ is the proper time of the particle multiplied by c, and a^μ is the four-potential of the field multiplied by q/mc^2. The metric four-tensor $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$.

For a planar field a^μ has y and z components that are functions of t and x. It is convenient to denote the transverse (two-vector) component of a^μ by $\mathbf{a}$, the transverse component of u^μ by $\mathbf{v}$, and the longitudinal components of u^μ by γ and u. In this notation, the transverse component of Eq. (1) is

$$d_\tau(\mathbf{v} + \mathbf{a}) = 0\,. \tag{2}$$

For a particle that is in front of the pulse initially, and is not moving transversely,

$$\mathbf{v} = -\,\mathbf{a}\,. \tag{3}$$

By using Eq. (3), one can rewrite the longitudinal components of Eq. (1) as

$$d_\tau\gamma = \partial_t\left(v^2/2\right), \quad d_\tau u = -\partial_x\left(v^2/2\right). \tag{4}$$

For a circularly polarized field

$$a^\mu = \left(0, 0, a\cos\phi, a\sin\phi\right)\big/\sqrt{2}\,. \tag{5}$$

We assume that the phase $\phi = t - sx$, where $s < 1$ is the inverse phase speed of the pulse, and the amplitude a is a function of $\psi = t - rx$, where $r > 1$ is the inverse group speed of the pulse. Equations (3)–(5) were solved in Ref. 1 for a particle that is at rest initially. The solution of these equations for a particle that is moving initially is similar. Since the ponderomotive potential $v^2/2$ is independent of ϕ, it follows from Eqs. (4) that

$$d_\tau(r\gamma - u) = 0\,. \tag{6}$$

By combining Eq. (6) with the definition of γ, one can show that

$$\gamma = \frac{r\left(r\gamma_0 - u_0\right) \mp \omega}{r^2 - 1}, \qquad u = \frac{\left(r\gamma_0 - u_0\right) \mp r\omega}{r^2 - 1}, \tag{7}$$

where

$$\omega = \left[\left(r\gamma_0 - u_0 \right)^2 - \left(r^2 - 1 \right)\left(1 + v^2 \right) \right]^{1/2}. \qquad (8)$$

In Eqs. (7) the – sign applies to the case in which $\gamma > ru$, which corresponds to a particle that is moving more slowly than the pulse, and the + sign applies to the case in which $\gamma < ru$, which corresponds to a particle that is moving more quickly than the pulse. By using the fact that $1 = \gamma_0^2 - u_0^2$, one can rewrite Eq. (8) in the convenient form

$$\omega = \left[\left(\gamma_0 - ru_0 \right)^2 - \left(r^2 - 1 \right) v^2 \right]^{1/2}. \qquad (9)$$

For the case in which $\gamma_0 = 1$ and $u_0 = 0$, Eqs. (7) and (9) reduce to the corresponding equations of Ref. 1. A particle that is moving more slowly than the pulse initially will be repelled by the pulse if $\omega = 0$. For this to happen the pulse intensity must equal the repelling intensity[1]

$$a^2 = 2\left(\gamma_0 - ru_0 \right)^2 \Big/ \left(r^2 - 1 \right), \qquad (10)$$

in which case the gain in particle energy[1]

$$\delta\gamma = 2\left(\gamma_0 - ru_0 \right) \Big/ \left(r^2 - 1 \right). \qquad (11)$$

For completeness, a covariant analysis of particle motion in a circularly polarized field is given in **Appendix A**, and a brief description of particle motion in an elliptically polarized field is given in **Appendix B**.

Equations (3), (7), and (9) define u^μ as a function of ψ. By combining the equation $d_\tau \psi = \gamma - ru$ with Eqs. (7), one can show that

$$d\tau/d\psi = \pm 1 \big/ \omega(\psi), \qquad (12)$$

where the + sign applies to the case in which $\gamma < ru$ and the – sign applies to the case in which $\gamma > ru$. If the solution of Eq. (12) can be inverted, u^μ can be expressed as an explicit function of τ.

To illustrate the particle motion we consider the simple profile

$$a(\psi) = e \sin\left(\pi\psi/2lr \right), \qquad (13)$$

where e^2 is the peak intensity of the pulse and l is its full-width at half-maximum. For this profile

$$\omega(\psi) = \left(\gamma_0 - ru_0 \right) \left[1 - m^2 \sin^2\left(\pi\psi/2lr \right) \right]^{1/2}, \qquad (14)$$

where

$$m^2 = \left(r^2 - 1 \right) e^2 \Big/ 2\left(\gamma_0 - ru_0 \right)^2 \qquad (15)$$

is the ratio of the pulse intensity to the repelling intensity.

When $m < 1$, the pulse overtakes the particle completely. In this case ψ varies between 0 and $2lr$, and the solution of Eq. (12) is

$$\tau(\psi) = \left[2lr/\pi\left(\gamma_0 - ru_0 \right) \right] F\left(\pi\psi/2lr, m \right), \qquad (16)$$

where F denotes the incomplete elliptic integral of the first kind, of modulus m.[5] It follows from Eqs. (7) and (9) that

$$t(\psi) = \frac{r\left(r\gamma_0 - u_0 \right) \tau(\psi) - \psi}{r^2 - 1},$$
$$x(\psi) = \frac{\left(r\gamma_0 - u_0 \right) \tau(\psi) - r\psi}{r^2 - 1}. \qquad (17)$$

The particle motion is illustrated in Fig. 70.52 for the case in which $\gamma_P = 30$, $\gamma_0 = 7$, and $e^2 = 7$ [The Lorentz factor γ_P is defined in the first of Eqs. (22)]. In Fig. 70.52(a) the phase, normalized to lr, is plotted as a function of time, normalized to $\gamma_P^2 l$. As the particle is accelerated by the front of the pulse, the rate of phase slippage decreases. However, since the peak intensity of the pulse is lower than the repelling intensity, the particle speed never equals the pulse speed and the pulse overtakes the particle. As the particle is decelerated by the back of the pulse, the rate of phase slippage increases. It is evident from Fig. 70.52(a) that the deceleration time equals the acceleration time. In Fig. 70.52(b) the longitudinal momentum is plotted as a function of the normalized time. Although the particle speed never exceeds the pulse speed, the energy associated with the transverse particle motion allows the particle momentum to exceed the pulse momentum. Because the longitudinal momentum is a symmetric function of time, the deceleration distance equals the acceleration distance.

When $m > 1$, the particle is repelled by the pulse. In this case ψ increases from 0 to $(2lr/\pi)\sin^{-1}(1/m)$ as the particle ascends the ponderomotive potential and decreases from $(2lr/\pi)\sin^{-1}(1/m)$ to 0 as the particle descends the ponderomotive potential. The solution of Eq. (12) is

$$\tau(\psi) = \begin{cases} \left[2lr/\pi m(\gamma_0 - ru_0)\right]F(\theta, 1/m), \\ \left[2lr/\pi m(\gamma_0 - ru_0)\right]\left[2K(1/m) - F(\theta, 1/m)\right], \end{cases} \tag{18}$$

where

$$\theta(\psi) = \sin^{-1}\left[m \sin(\pi\psi/2lr)\right] \tag{19}$$

and K denotes the complete elliptic integral of the first kind, of modulus m.[5] The first form of Eq. (18) applies to the ascent and the second form applies to the descent. Equations (17) apply to both the ascent and descent, provided that τ is defined by Eqs. (18) and (19). The particle motion is illustrated in Fig. 70.53 for the case in which $\gamma_P = 30$, $\gamma_0 = 7$, and $e^2 = 10$. In Fig. 70.53(a) the normalized phase is plotted as a function of the normalized time. Initially, the pulse overtakes the particle and the rate of phase slippage is positive. Since the peak intensity of the pulse is higher than the critical intensity, the particle is accelerated until its speed equals the pulse speed and the rate of phase slippage is zero. Subsequently, the particle overtakes the pulse and the rate of phase slippage is negative. The descent time is longer than the ascent time because the time dilation associated with a particle moving faster than the pulse is larger than that associated with a particle moving slower than the pulse. In Fig. 70.53(b) the longitudinal momentum is plotted as a function of the normal-

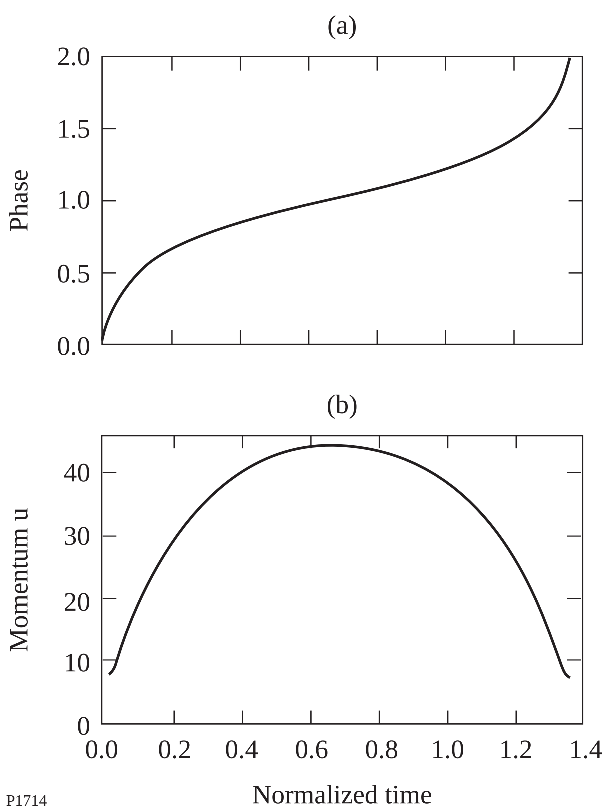

Figure 70.52
Particle trajectory for the case in which $\gamma_P = 30$ and $\gamma_0 = 7$. The pulse intensity $e^2 = 7$ is slightly lower than the repelling intensity [Eq. (10)]. (a) Normalized phase ψ/lr plotted as a function of the normalized time $t/\gamma_P^2 l$; (b) longitudinal momentum u plotted as a function of the normalized time.

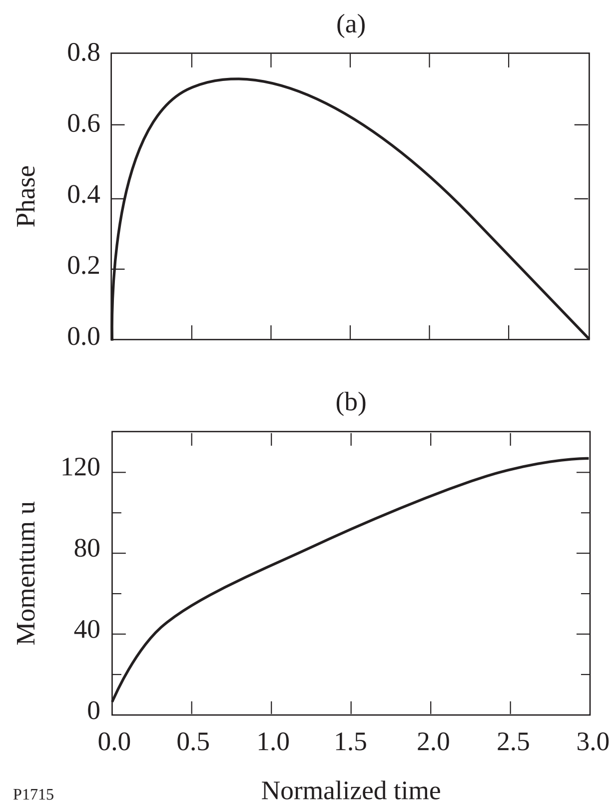

Figure 70.53
Particle trajectory for the case in which $\gamma_P = 30$ and $\gamma_0 = 10$. The pulse intensity $e^2 = 10$ is slightly higher than the repelling intensity [Eq. (10)]. (a) Normalized phase ψ/lr plotted as a function of the normalized time $t/\gamma_P^2 l$; (b) longitudinal momentum u plotted as a function of the normalized time.

ized time. Since the particle does not reach the peak of the pulse, the x component of the ponderomotive force is always positive and the longitudinal momentum of the particle increases monotonically. Because the longitudinal momentum is an asymmetric function of time, the descent distance is longer than the ascent distance.

Dephasing Time of an Accelerated Particle

The previous analysis shows how an intense pulse repels a charged particle that is in front of the pulse. The relation between the pulse intensity, the particle injection energy, and the gain in particle energy was studied in Ref. 1. In this section the time required for the pulse to catch and repel the particle, and, subsequently, for the particle to outrun the pulse, is studied. This time is referred to as the dephasing time and is denoted by T. The distance traveled by the particle during the dephasing time is referred to as the dephasing distance and is denoted by X. It follows from Eqs. (17) and (18) that

$$T = \frac{4lr^2\left(r\gamma_0 - u_0\right)K\left(1/m\right)}{\pi m\left(r^2-1\right)\left(\gamma_0 - ru_0\right)} \tag{20}$$

and $X = T/r$. For future reference, notice that $K(1/m) \rightarrow \log\,[4/(m^2-1)^{1/2}]$ as $m \rightarrow 1$ and $K\,(1/m) \rightarrow \pi/2$ as $m \rightarrow \infty$.[5]

Formula (20) for the dephasing time exhibits a complicated dependence on the pulse intensity and speed and the initial particle momentum. One can gain insight into the underlying physics by performing a pulse-frame analysis of the acceleration process. In the notation of Ref. 1, $'$ denotes a pulse-frame quantity, the subscript A denotes the initial position of the particle, and B denotes the position at which the particle is repelled.

The pulse-frame energy and momentum of the particle are related to the laboratory-frame energy and momentum by the equations

$$\gamma' = \gamma_P\gamma - u_P u, \quad u' = \gamma_P u - u_P\gamma, \tag{21}$$

where

$$\gamma_P = r\Big/\left(r^2-1\right)^{1/2}, \quad u_P = 1\Big/\left(r^2-1\right)^{1/2}. \tag{22}$$

In these equations γ_P is the Lorentz factor associated with the pulse speed $1/r$ and $u_P = \left(\gamma_P^2 - 1\right)^{1/2}$. If one uses the linear group speed of the pulse to estimate the Lorentz factor, $\gamma_P = \omega_0/\omega_e$, where ω_0 is the carrier frequency of the pulse and ω_e is the electron-plasma frequency.[6]

In the pulse frame v^2 is time-independent. It follows from the first of Eqs. (4) that γ' is constant and, hence, that $\left(u'\right)^2 + v^2 = \left(u'_A\right)^2$. Since $dx'/dt' = u'/\gamma'_A$, it follows that

$$T' = 2\gamma'_A\int_{x'_B}^{x'_A} \frac{dx'}{\left[\left(u'_A\right)^2 - v^2\left(x'\right)\right]^{1/2}}. \tag{23}$$

In Eq. (23) the factor of 2 arises because the pulse-frame descent time equals the pulse-frame ascent time. The factor of γ'_a arises because of the difference between proper time and pulse-frame time. Provided one ignores the distinction between momentum and velocity, the integral in Eq. (23) represents the ascent time of a nonrelativistic particle in the potential well $v^2\left(x'\right)/2$. In the pulse frame $a = e\sin\left(-\pi x'/2l'\right)$, where $l' = \gamma_P l$. For this profile

$$T' = 2\gamma'_A\left(2l'/\pi\right)\left(\sqrt{2}/e\right)K\left(\sqrt{2}u'_A/e\right). \tag{24}$$

The factor of $2l'/\pi$ arises because the ponderomotive force associated with the pulse is inversely proportional to the pulse length. Although Eq. (24) is complicated, the origin of each factor is well understood.

In the pulse frame the particle begins and ends its interaction with the pulse at point A. Since $X' = 0$, it follows that

$$T = \gamma_P T', \quad X = u_P T'. \tag{25}$$

Notice that $X = T/r$, as stated after Eq. (20). It follows from Eqs. (21) and (22) that $r\left(r\gamma_0 - u_0\right)\Big/\left(r^2-1\right) = \gamma_P\gamma'_A$ and $4lr/\pi m\left(\gamma_0 - ru_0\right) = 2\left(2l'/\pi\right)\left(\sqrt{2}/e\right)$. Thus, Eq. (24) and the first of Eqs. (25) agree with Eq. (20).

It is convenient to define the normalized dephasing time

$$\overline{T} = \left(4\sqrt{2}\,\gamma'_A \;\; \pi e\right)K\left(\sqrt{2}\,u'_A/e\right), \tag{26}$$

which is the dephasing time divided by $\gamma_P^2 l$. The factor of l was due to the inverse dependence of the ponderomotive force on the pulse length. One factor of γ_P was due to the Lorentz transformation of the pulse length from the laboratory frame to the pulse frame; the other factor was due to the Lorentz transformation of the dephasing time from the pulse frame to the laboratory frame. These factors do not depend on the physical origin or shape of the potential well in which the particle moves. Thus, it was inevitable that they should be the same as the factors that control the dephasing time of an electron in the laser beat-wave accelerator[7] or the laser wake-field accelerator.[7,8] For completeness, a brief analysis of the particle motion and dephasing time associated with these indirect acceleration schemes is given in **Appendix C**.

Just as a pulse-frame analysis of the acceleration process fosters insight into the dephasing time, so also does it foster insight into the energy gain. In the pulse frame the particle energy is constant, and the final particle momentum has the same magnitude as the initial particle momentum and the opposite sign: $\delta\gamma' = 0$ and $\delta u' = 2|u'_A|$. It follows from these results and Eqs. (21) that

$$\delta\gamma = 2u_P|u'_A|, \tag{27}$$

in agreement with Eq. (11).

The normalized dephasing time is plotted as a function of pulse intensity in Fig. 70.54 for the case in which $\gamma_P = 30$. In Fig. 70.54(a) the injection energy $\gamma_A = 7$. The solid line denotes the exact dephasing time [Eq. (26)], and the broken line denotes the approximate dephasing time $2\sqrt{2}\gamma'_A/e$. For the chosen values of γ_P and γ_A the approximate dephasing time is $6.4/e$. When the pulse intensity is close to the repelling intensity, the particle lingers near the peak of the pulse and the dephasing time is long. As the pulse intensity increases, point B moves toward the front of the pulse and the dephasing time decreases. In the high-intensity regime this decrease is gradual. Since the pulse energy located behind point B is wasted, there is little to be gained by using pulse intensities that exceed the critical intensity by more than a factor of 2. Since the injection energy is constant, so also is the energy gain [Eq. (27)]. In Fig. 70.54(b) the injection energy

$$\gamma_A = \gamma_P\mu - \left[\left(\gamma_P^2 - 1\right)\left(\mu^2 - 1\right)\right]^{1/2}, \tag{28}$$

where $\mu = \left(1 + e^2/4\right)^{1/2}$ is a measure of the pulse intensity. This choice of injection energy ensures that the repelling intensity is one-half of the pulse intensity. For this injection energy $u'_A = -e/2$, $\gamma'_A = \left(1 + e^2/4\right)^{1/2}$, and the saturation time is $1.7\left(1 + 4/e^2\right)^{1/2}$, independent of γ_P. In the low-intensity regime the dephasing time is long because the ponderomotive force is weak. In the high-intensity regime the dephasing time is almost independent of pulse intensity because the increase in ponderomotive force that accompanies an increase in pulse intensity is offset by the corresponding decrease in injection energy. It follows from Eq. (27) and the preceding discussion that the energy gain equals $u_P e$. As the pulse intensity increases, the energy gain increases and the required injection energy decreases.

The normalized dephasing time is plotted as a function of injection energy in Fig. 70.55 for the case in which $\gamma_P = 30$. In Fig. 70.55(a) the pulse intensity $e^2 = 10$. The solid line denotes the exact dephasing time [Eq. (26)], and the broken line denotes the approximate dephasing time $2\sqrt{2}\gamma'_A/e$. For the chosen values of γ_P and e the approximate dephasing time is

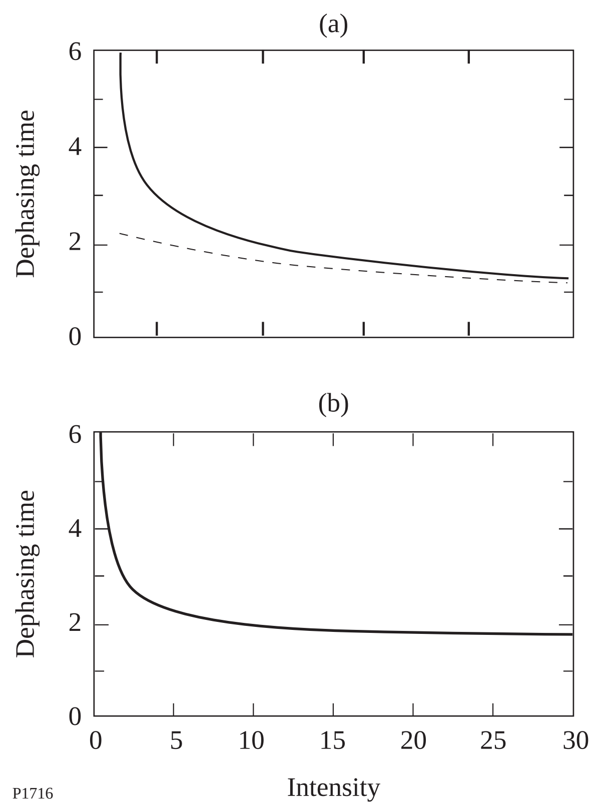

Figure 70.54
Normalized dephasing time [Eq. (26)] plotted as a function of pulse intensity for the case in which γ_P = 30. (a) The particle injection energy γ_A = 7. (b) The particle injection energy [Eq. (28)] ensures that the repelling intensity [Eq. (10)] is one-half of the pulse intensity.

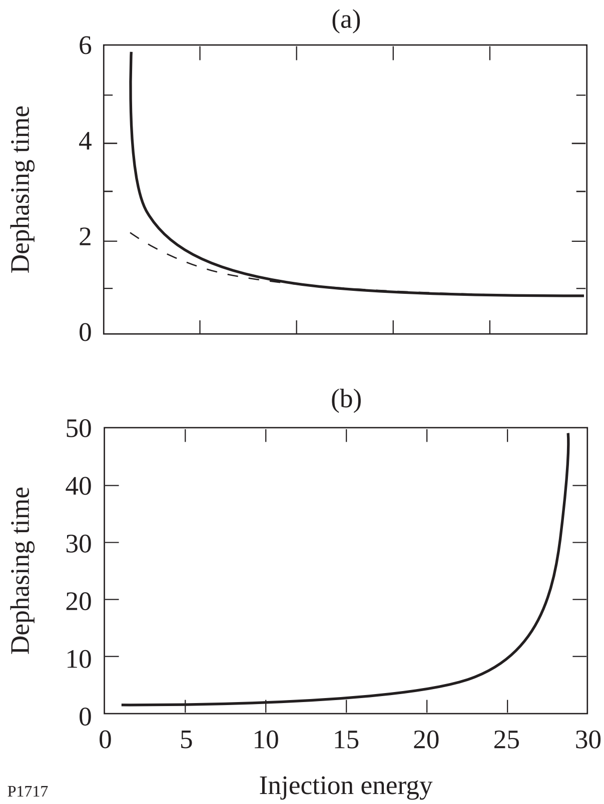

Figure 70.55
Normalized dephasing time [Eq. (26)] plotted as a function of particle injection energy for the case in which $\gamma_P = 30$. (a) The pulse intensity $e^2 = 10$. (b) The pulse intensity [Eq. (29)] is twice the repelling intensity [Eq. (10)].

$0.89\,\gamma'_A$. When the injection energy is close to the repelling energy, the particle lingers near the peak of the pulse and the dephasing time is long. As the injection energy increases, point B moves toward the front of the pulse and the dephasing time decreases. In the high-energy regime the dephasing time is almost independent of the injection energy because $\gamma'_A \approx 1$. The energy gain decreases as the injection energy increases. In Fig. 70.55(b) the pulse intensity

$$e^2 = 4\left(u'_A\right)^2 \qquad (29)$$

is twice the repelling intensity and the dephasing time is $1.7\,\gamma'_A/\left|u'_A\right|$. In the low-energy regime the dephasing time is almost independent of the injection energy because $\gamma'_A \approx \gamma_P$ and $u'_A \approx -\gamma_P$. The ratio $\gamma'_A/\left|u'_A\right|$ is almost independent of γ_P. In the high-energy regime the dephasing time is long and the energy gain is small because $\gamma'_A \approx 1$ and $\left|u'_A\right| << 1$.

Summary

The motion of an electron in the electromagnetic field associated with a circularly polarized laser pulse of infinite width was studied analytically. When the pulse intensity is lower than the repelling intensity [Eq. (10)], the pulse overtakes the electron completely. When the pulse intensity is higher than the repelling intensity, the electron is repelled by the pulse and eventually outruns it. The time taken for the electron to outrun the pulse is called the dephasing time and is the product of two terms. The first term is $\gamma_P^2 l$, where γ_P is the Lorentz factor associated with the pulse speed and l is the pulse length. The second term [Eq. (26)] depends on the pulse intensity, the pulse shape, and the electron injection energy. As a rough guideline, the second term is of order unity unless the pulse intensity is close to the repelling intensity. For a pulse of finite width, an electron that is not close to the pulse axis initially will be expelled from the pulse by the radial component of the ponderomotive force.[9] Further work is needed to quantify this snowplow effect.

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Appendix A: Covariant Analysis of the Particle Motion in a Planar Field

The resolution of Eq. (1) into longitudinal and transverse components is facilitated by the introduction of the four-vector k^μ, which is defined by the equation $\psi = k^\nu x_\nu$, and the four-vector l^μ, which is defined by the equations $l^\nu l_\nu = -k^\nu k_\nu$, $l^\nu k_\nu = 0$, and $l^\nu a_\nu = 0$, where a^μ is the transverse four-potential of a planar field of arbitrary polarization. In the laboratory frame $k^\mu = (1,r,0,0)$ and $l^\mu = (r,1,0,0)$. By using these four-vectors, one can write

$$x_\mu = y_\mu + \psi k_\mu / k^\nu k_\nu + \theta l_\mu / l^\nu l_\nu\,, \qquad \text{(A1)}$$

where y_μ is transverse and $\theta = l^\nu x_\nu$. In a similar way, one can write

$$u_\mu = v_\mu + \left(k^\nu u_\nu\right) k_\mu / k^\nu k_\nu + \left(l^\nu u_\nu\right) l_\mu / l^\nu l_\nu\,, \qquad \text{(A2)}$$

where $v_\mu = dy_\mu/d\tau$ is transverse, $k^\nu u_\nu = d_\tau\psi$, and $l^\nu u_\nu = d_\tau\theta$.

The transverse component of Eq. (1) is

$$d_\tau\left(v_\mu + a_\mu\right) = 0, \qquad \text{(A3)}$$

from which it follows that

$$v_\mu(\tau) = v_\mu(0) + a_\mu(0) - a_\mu(\tau). \qquad \text{(A4)}$$

Equation (A4) is the analog of Eq. (3).

By using Eq. (A4), one can rewrite the right side of Eq. (1) as $-\partial_\mu\left(v^\nu v_\nu/2\right)$. Since $v^\nu v_\nu$ was assumed to be a function of ψ, $\partial_\mu = k_\mu d_\psi$. It follows from these results that the longitudinal components of Eq. (1) are

$$d_\tau\left(k^\mu u_\mu\right) = -k^\mu k_\mu d_\psi\left(v^\nu v_\nu/2\right), \quad d_\tau\left(l^\mu u_\mu\right) = 0. \qquad \text{(A5)}$$

It follows from the second of Eqs. (A5) that

$$l^\mu u_\mu(\tau) = l^\mu u_\mu(0). \qquad \text{(A6)}$$

One way to obtain an expression for $k^\nu u_\nu$ is to use the identity $u^\nu u_\nu = 1$, which can be rewritten as

$$v^\nu v_\nu + \left(k^\nu u_\nu\right)^2 \big/ k^\nu k_\nu + \left(l^\nu u_\nu\right)^2 \big/ l^\nu l_\nu = 1. \qquad \text{(A7)}$$

It follows from Eq. (A7) that

$$\left[k^\nu u_\nu(\tau)\right]^2 = \left[l^\nu u_\nu(\tau)\right]^2 + k^\nu k_\nu\left[1 - v^\nu v_\nu(\tau)\right]. \qquad \text{(A8)}$$

Equations (A6) and (A8) are the analogs of Eqs. (7) and (8). By using the expression for $k^\nu u_\nu(0)$ that follows from Eq. (A8), and Eq. (A6), one can show that

$$\left[k^\nu u_\nu(\tau)\right]^2 = \left[k^\nu u_\nu(0)\right]^2 + k^\nu k_\nu\left[v^\nu v_\nu(0) - v^\nu v_\nu(\tau)\right]. \qquad \text{(A9)}$$

Equation (A9) is the analog of Eq. (9). Another way to obtain an expression for $k^\nu u_\nu$ is to change the independent variable in the first of Eqs. (A5) from τ to ψ. Since $d_\tau\psi = k^\nu u_\nu$, the first of Eqs. (A5) becomes

$$d_\psi\left[\left(k^\nu u_\nu\right)^2 \big/ 2\right] = -k^\mu k_\mu d_\psi\left(v^\nu v_\nu/2\right), \qquad \text{(A10)}$$

from which Eq. (A9) follows.

Finally, since $v^\nu v_\nu$ is a function of ψ, Eqs. (A4), (A6), and (A9) express u_μ as a function of ψ. To express u_μ as a function of τ one must invert the solution of the phase equation

$$d\tau/d\psi = \pm 1/\omega(\psi), \qquad \text{(A11)}$$

where ω is the square root of the terms on the right side of Eq. (A9).

Appendix B: Guiding-Center Motion in a Planar Field

Equation (6) is valid when the v^2 terms in Eqs. (4) are independent of ϕ. To satisfy this condition we assumed that the field is circularly polarized and that the particle is in front of the pulse initially and not moving transversely. Equations (7), (9), and (10) follow from Eq. (6) and the definition of γ, which requires that

$$d_\tau\left(\gamma^2 - u^2 - v^2\right) = 0. \qquad \text{(B1)}$$

For the elliptically polarized field

$$a^\mu = \left(0, 0, a_y \cos\phi, a_z \cos\phi\right), \qquad \text{(B2)}$$

where $a_y = a\delta$ and $a_z = a\left(1 - \delta^2\right)^{1/2}$, the v^2 terms in Eqs. (4) are not independent of ϕ, and Eq. (6) is not valid. However, the particle motion is known to consist of a fast oscillation about a guiding center and a guiding-center drift that varies slowly. In a vacuum, the guiding-center motion is governed by the equation[9]

$$d_\tau\left\langle u_\mu\right\rangle = -\partial_\mu\left\langle a^\nu a_\nu\right\rangle/2, \qquad \text{(B3)}$$

where $\langle\ \rangle$ denotes a ϕ-average and $\left\langle a^\nu a_\nu\right\rangle = -a^2/2$. We expect Eq. (B3) to provide a reasonable description of the guiding-center motion in a rarefied plasma, in which the phase speed of the field is slightly higher than the speed of light. Equation (B3) has associated with it the conservation equation

$$d_\tau\left(\left\langle u^\nu\right\rangle\left\langle u_\nu\right\rangle + \left\langle a^\nu a_\nu\right\rangle\right) = 0. \qquad \text{(B4)}$$

Since the ponderomotive potential $a^2/4$ is independent of ϕ, it follows from Eq. (B3) that

$$d_\tau\left(r\langle\gamma\rangle - \langle u\rangle\right) = 0. \tag{B5}$$

Equation (B5) is the analog of Eq. (6). Since $\langle \mathbf{v}\rangle$ is constant, Eq. (B4) reduces to

$$d_\tau\left(\langle\gamma\rangle^2 - \langle u\rangle^2 - a^2/2\right) = 0. \tag{B6}$$

Equation (B6) is the analog of Eq. (B1). Thus, for a particle that is in front of the pulse initially, $\langle\gamma\rangle$ and $\langle u\rangle$ are given by Eqs. (7) and (9), in which v^2 is replaced by $a^2/2$, and the repelling conditions are described by Eq. (10).

Appendix C: Particle Motion in a Planar Electrostatic Field

The four-potential of an electrostatic field can be written as

$$a_\mu = pk_\mu / k^\nu k_\nu + ql_\mu / l^\nu l_\nu, \tag{C1}$$

where k^μ and l^μ were defined in **Appendix A**. We assume that a_μ is a function of ψ, from which it follows that $\partial_\mu = k_\mu d_\psi$. Since the electrostatic field is unaffected by the gauge transformation $a_\mu \to a_\mu + \partial_\mu b$, where b is an arbitrary function of ψ, p is redundant. In the Lorentz gauge $p = 0$.

By substituting decomposition (C1) in Eq. (1) and contracting the resulting equation with k^μ, one can show that

$$d_\tau\left(k^\mu u_\mu + p\right) = \left(k^\nu u_\nu\right) d_\psi p - \left(l^\nu u_\nu\right) d_\psi q. \tag{C2}$$

Since $k^\nu u_\nu = d_\tau \psi$, the p terms in Eq. (C2) cancel, as they must do. By substituting decomposition (C1) in Eq. (1) and contracting the resulting equation with l^μ, one can show that

$$d_\tau\left(l^\mu u_\mu + q\right) = 0. \tag{C3}$$

It follows from Eq. (C3) that

$$l^\mu u_\mu(\tau) = l^\mu u_\mu(0) + q(0) - q(\tau). \tag{C4}$$

One way to obtain an expression for $k^\nu u_\nu$ is to use the identity $u^\nu u_\nu = 1$, which can be rewritten as

$$\left[k^\nu u_\nu(\tau)\right]^2 = \left[l^\nu u_\nu(\tau)\right]^2 - l^\nu l_\nu. \tag{C5}$$

Another way to obtain an expression for $k^\nu u_\nu$ is to solve Eq. (C2) directly. By changing the independent variable from τ to ψ, one can rewrite Eq. (C2) as

$$d_\psi\left[\left(k^\nu u_\nu\right)^2/2\right] = d_\psi\left[\left(l^\nu u_\nu\right)^2/2\right], \tag{C6}$$

from which Eq. (C5) follows.

Since q is a function of ψ, Eqs. (C4) and (C5) express u_μ as a function of ψ. To express u_μ as a function of τ, one must invert the solution of the phase equation

$$d\tau/d\psi = \pm 1/\omega(\psi), \tag{C7}$$

where ω is the square root of the terms on the right side of Eq. (C5).

In the wave frame $k'_\mu = (0, l, 0, 0)$ and $l'_\mu = (l, 0, 0, 0)$, where $l = \left(r^2 - 1\right)^{1/2}$. It follows from these results that $\psi = -lx'$, $k^\nu u_\nu = -lu'$, $l^\nu u_\nu = l\gamma'$, and $q = l\phi'$, where ϕ is the electrostatic potential. Thus, Eq. (C4) can be rewritten as

$$\gamma'(\tau) = \gamma'(0) + \phi'(0) - \phi'(\tau) = 0, \tag{C8}$$

Eq. (C5) can be rewritten as

$$\left[u'(\tau)\right]^2 = \left[\gamma'(\tau)\right]^2 - 1, \tag{C9}$$

and Eq. (C7) can be rewritten as

$$d\tau/dx' = \mp 1/\omega(x'), \tag{C10}$$

where ω is the square root of the terms on the right side of Eq. (C9). The dephasing time of an accelerated particle can be determined from Eq. (C10) in a manner similar to that described previously. In particular, by considering the relations between laboratory-frame and wave-frame quantities, one can show that the dephasing time is proportional to $\gamma_W^2\lambda$, where γ_W is the Lorentz factor associated with the phase speed of the wave and λ is the wavelength.

The potential associated with a large-amplitude plasma wave is described by elliptic functions. Simple formulas for the injection energy and energy gain associated with this potential were determined by Esarey and Piloff.[10] The dephasing times associated with this and other potentials were studied by Teychenné, Bonnaud, and Bobin.[11,12]

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