Multiple Scale Derivation of the Relativistic Ponderomotive Force

The ponderomotive force associated with a light wave of variable amplitude[1–10] drives many phenomena that occur in inertial confinement fusion[11] and particle acceleration[12] experiments. The existing formula for the ponderomotive force was derived under the assumption that the quiver speed of electrons oscillating in the applied electric field is much less than the speed of light. With the advent of intense laser pulses,[13] it is important to extend this formula to electron quiver speeds that are comparable to the speed of light.

As an introduction to this subject, we review the derivation of the ponderomotive term in the electron-fluid momentum equation. The standard form of this equation is

$$\left(\partial_t + \mathbf{v}\cdot\nabla\right)\left(\gamma\,\mathbf{v}\right) = -(\mathbf{E} + \mathbf{v}\times\mathbf{B}), \tag{1}$$

where

$$\gamma = \left(1 - v^2\right)^{-1/2} \tag{2}$$

is the Lorentz factor associated with the fluid velocity and

$$\mathbf{E} = -\partial_t \mathbf{A}, \quad \mathbf{B} = \nabla\times\mathbf{A} \tag{3}$$

in the radiation gauge. These differ from the usual equations in that $\omega t \rightarrow t$, $k\mathbf{x} \rightarrow \mathbf{x}$, $\mathbf{v}/c \rightarrow \mathbf{v}$, $e\mathbf{E}/m\omega c \rightarrow \mathbf{E}$, $e\mathbf{B}/m\omega c \rightarrow \mathbf{B}$, and $e\mathbf{A}/mc^2 \rightarrow \mathbf{A}$.

By using the vector identity[14]

$$(\mathbf{v}\cdot\nabla)(\gamma\mathbf{v}) = \nabla\gamma - \mathbf{v}\times\left[\nabla\times(\gamma\mathbf{v})\right], \tag{4}$$

one can rewrite the momentum equation as

$$\partial_t\left(\gamma\,\mathbf{v} - \mathbf{A}\right) = \mathbf{v}\times\left[\nabla\times\left(\gamma\,\mathbf{v} - \mathbf{A}\right)\right] - \nabla\gamma, \tag{5}$$

from which follows the relativistic vorticity equation

$$\partial_t\left[\nabla\times\left(\gamma\,\mathbf{v} - \mathbf{A}\right)\right] = \nabla\times\left\{\mathbf{v}\times\left[\nabla\times\left(\gamma\,\mathbf{v} - \mathbf{A}\right)\right]\right\}. \tag{6}$$

For a plasma that is at rest before the laser pulse arrives, $\nabla\times\left(\gamma\,\mathbf{v} - \mathbf{A}\right) = 0$ initially. Equation (6) ensures that $\nabla\times\left(\gamma\,\mathbf{v} - \mathbf{A}\right) = 0$ for all time. Thus, the momentum equation can be rewritten as[14]

$$\partial_t\left(\mathbf{u} - \mathbf{A}\right) = -\nabla\gamma, \tag{7}$$

where the fluid momentum $\mathbf{u} = \gamma\,\mathbf{v}$. It follows from this definition that $\gamma = \left(1 + u^2\right)^{1/2}$.

The ponderomotive term on the right side of Eq. (7) is valid for arbitrary laser intensity. Together with the continuity and Maxwell equations, it allows one to analyze the interaction of a laser pulse with an electron fluid. However, there is a tradition in plasma physics of looking at the same phenomenon from different viewpoints. By doing so, one often gains physical insight into the phenomenon under study. The ponderomotive term in Eq. (7) is not the force on a Lagrangian fluid element or a single electron. Consequently, it cannot be used as the foundation of a single-particle or kinetic analysis of the interaction of a laser pulse with a plasma.

In the following sections we present (1) an analytical study of the motion of an electron in a light wave of constant amplitude; (2) using the results of this study, a heuristic derivation of the formula for the ponderomotive force associated with a light wave of variable amplitude; (3) numerical and analytical verification of this formula; and, finally, (4) a summary of the results.

Particle Motion in a Plane Wave

The motion of a charged particle, of charge q and mass m, in an electromagnetic field is governed by the equation[15]

$$d_\tau\left(u_\mu + a_\mu\right) = u^\nu \partial_\mu a_\nu\,, \tag{8}$$

where τ is the proper time of the particle multiplied by c, u^μ is the four-velocity of the particle divided by c, a^μ is the four-potential of the field multiplied by q/mc^2 and $\partial_\mu = \partial/\partial x^\mu$. For an elliptically polarized field

$$a^\mu = \left(0, 0, e_y \cos\phi, e_z \sin\phi\right), \tag{9}$$

where $e_y = e\delta$, $e_z = e\left(1-\delta^2\right)^{1/2}$, and $\phi = t{-}x$.

The motion of a charged particle in a plane wave is well known.[16–19] We present an analysis of this motion here because it is the foundation of analyses presented later in this article. Since the four-potential does not depend on y or z, it follows from Eq. (8) that

$$d_\tau\left(\mathbf{u}_\perp + \mathbf{a}_\perp\right) = 0\,. \tag{10}$$

Transverse canonical momentum is conserved. It follows from Eq. (10) that

$$\mathbf{u}_\perp(\tau) = \mathbf{u}_\perp(0) + \mathbf{a}_\perp(0) - \mathbf{a}_\perp(\tau)\,. \tag{11}$$

The t and x components of Eq. (8) are

$$d_\tau \gamma = \tfrac{1}{2}\partial_t u_\perp^2\,, \quad d_\tau u_x = -\tfrac{1}{2}\partial_x u_\perp^2\,. \tag{12}$$

Since the four-potential is a function of $t{-}x$, it follows from Eqs. (12) that

$$d_\tau\left(\gamma - u_\parallel\right) = 0\,. \tag{13}$$

Because the particle gains energy and momentum at the expense of the field, the ratio of particle momentum to particle kinetic energy is identical to the ratio of field momentum to field energy, which is 1 in the units of Eq. (8). By combining Eq. (13) with the definition of γ, one can show that

$$u_\parallel(\tau) = u_\parallel(0) + \frac{u_\perp^2(\tau) - u_\perp^2(0)}{2\left[\gamma(0) - u_\parallel(0)\right]}\,. \tag{14}$$

The corresponding equation for $\gamma(t)$ follows from Eqs. (13) and (14). Because the transverse potential $\mathbf{a}_\perp$ is a function of ϕ rather than τ, Eqs. (11) and (13) describe the particle momentum implicitly. One can make this description explicit and determine the particle trajectory $x^\mu(\tau)$ by using the result

$$d_\tau \phi = \gamma(0) - u_\parallel(0)\,. \tag{15}$$

The proper frequency of the wave is constant.

It is clear from Eqs. (11), (14), and (15) that the particle motion is a superposition of sinusoidal oscillations in τ and steady drifts in τ. It follows from Eq. (11) that the transverse drifts are given by

$$\begin{aligned} \langle u_y \rangle &= u_y(0) + e_y \cos\left(-x_0\right), \\ \langle u_z \rangle &= u_z(0) + e_z \sin\left(-x_0\right), \end{aligned} \tag{16}$$

where $\langle\,.\,\rangle$ denotes the τ-average $\int_0^{2\pi} .\; d\tau/2\pi$ and $(x_0,0,0)$ is the initial position of the particle. By decomposing the longitudinal momentum into its oscillatory component

$$u_\parallel(\tau) - \langle u_\parallel \rangle = \frac{u_\perp^2(\tau) - \langle u_\perp^2 \rangle}{2\left[\gamma(0) - u_\parallel(0)\right]} \tag{17}$$

and its drift component

$$\langle u_\parallel \rangle = u_\parallel(0) + \frac{\langle u_\perp^2 \rangle - u_\perp^2(0)}{2\left[\gamma(0) - u_\parallel(0)\right]}, \tag{18}$$

and combining Eqs. (11) and (18), one can show that the longitudinal drift is given by

$$\begin{aligned} \langle u_x \rangle = u_x(0) + \Big[& 4\langle u_y \rangle e_y \cos\left(-x_0\right) \\ & + 4\langle u_z \rangle e_z \sin\left(-x_0\right) - e_y^2 \cos\left(-2x_0\right) \\ & + e_z^2 \cos\left(-2x_0\right)\Big] \Big/ 4\left[\gamma(0) - u_x(0)\right]. \end{aligned} \tag{19}$$

For linear polarization Eq. (19) reduces to

$$\begin{aligned} \langle u_x \rangle = u_x(0) + \Big[& 4\langle u_y \rangle e \cos\left(-x_0\right) \\ & - e^2 \cos\left(-2x_0\right)\Big] \Big/ 4\left[\gamma(0) - u_x(0)\right], \end{aligned} \tag{20}$$

whereas for circular polarization it reduces to

$$\langle u_x \rangle = u_x(0) + e\Big[\langle u_y \rangle \cos(-x_0) + \langle u_z \rangle \sin(-x_0)\Big]\Big/\sqrt{2}\,[\gamma(0) - u_x(0)]. \qquad (21)$$

The corresponding equations for $\langle \gamma \rangle$ follow from Eq. (13) and Eqs. (19)–(21).

For completeness, a covariant analysis of the particle motion is given in **Appendix A**.

Heuristic Derivation of the Ponderomotive Force

The method used to solve Eq. (8) for a plane wave of constant amplitude can also be used when the wave amplitude *e* is a function of *t*–*x*. In fact, Eqs. (11), (14), and (15) are still valid. When the wave amplitude varies slowly compared to the wave phase, the particle motion consists of an oscillation about a guiding center and a guiding-center drift that varies slowly. As the guiding center drifts, the oscillation amplitude follows the wave amplitude at the guiding center adiabatically.

To describe this motion quantitatively, let ξ^μ be the position four-vector of the guiding center and $\upsilon^\mu = d_\tau \xi^\mu$ be the associated four-momentum. The ponderomotive four-force is the proper rate of change of the guiding-center four-momentum. One might expect this four-force to also be the average rate of change of the particle four-momentum. However, by averaging the transverse particle motion, one finds that

$$\begin{aligned} \langle d_\tau u_y \rangle &\approx \big[d_{\tau_0} e_y(\tau_0)\big]\cos(\tau_0), \\ \langle d_\tau u_z \rangle &\approx \big[d_{\tau_0} e_z(\tau_0)\big]\sin(\tau_0), \end{aligned} \qquad (22)$$

where τ_0 is the initial phase with respect to which the average is taken. Because the oscillation amplitude changes during each oscillation, the transverse components of the momentum change by amounts that depend on the initial phase. However, it follows from Eq. (11) that the transverse components of the guiding-center momentum are constant. Thus, if one is to determine the ponderomotive four-force by averaging, one must discount terms that depend on the initial phase. With this caveat added to the definition of $\langle\,.\,\rangle$, one can write

$$d_\tau \upsilon_y = \langle d_\tau u_y \rangle \approx 0, \quad d_\tau \upsilon_z = \langle d_\tau u_z \rangle \approx 0 \qquad (23)$$

and show that

$$d_\tau \upsilon_x = \langle d_\tau u_x \rangle \approx d_\tau\big(e_y^2 + e_z^2\big)\big/4\big[\gamma(0) - u_\parallel(0)\big]. \qquad (24)$$

By using the relationship between *x* and ϕ, and Eq. (15), one can show that $d_\tau = -\big[\gamma(0) - u_\parallel(0)\big]\partial_x$. It follows from this result and Eq. (24) that

$$d_\tau \upsilon_x \approx -\partial_x\big(e^2/4\big). \qquad (25)$$

In a similar way, one can show that

$$d_\tau \upsilon_t \approx \partial_t\big(e^2/4\big). \qquad (26)$$

By using the facts that $e^2/2 = \langle a_\perp^2 \rangle$ and $a_\perp^2 = -a_\nu a^\nu$, one can rewrite Eqs. (23), (25), and (26) as

$$d_\tau \upsilon_\mu \approx -\partial_\mu \langle a_\nu a^\nu/2 \rangle. \qquad (27)$$

The second term in this relation is the ponderomotive four-force.

The guiding-center Eq. (27) was derived for the special case in which *e* is a function of *t*–*x*. However, the principle of Lorentz covariance suggests that it is valid for the general case in which *e* is a function of *t*, *x*, *y*, and *z*. Consequently, we postulate that[20]

$$d_{\tau\tau}^2 \xi_\mu = -\partial_\mu \langle a_\nu a^\nu/2 \rangle\big|_{\xi_\mu} \qquad (28)$$

and the initial guiding-center momentum in a wave of variable amplitude is identical to the particle drift momentum in a wave of constant amplitude, which is given by Eqs. (16) and (19). For future reference, Eq. (28) has associated with it the conservation equation

$$d_\tau\big(\upsilon_\mu \upsilon^\mu/2 + \langle a_\nu a^\nu/2 \rangle\big) = 0. \qquad (29)$$

Numerical Study of the Particle Motion

To test the guiding-center model described in the previous section, we studied three representative examples numerically. The first example concerns a particle that moves in front of a laser pulse. We considered a wide, circularly polarized pulse, with $e = 3\sin^2\left[0.05(t-x)\right]$, and chose $u_x(0) = 1$, $u_y(0) = 1$, and $u_z(0) = 1$. Because the pulse propagates at the speed of light, it overtakes the particle. The resulting particle motion is illustrated in Figs. 69.25 and 69.26, in which the solid lines denote the particle trajectory, determined numerically from Eq. (8) and the initial conditions, and the dashed lines denote the guiding-center trajectory, determined numerically from Eqs. (28), (16), and (19). As the pulse overtakes the particle, the amplitudes of the transverse components of the oscillation increase and decrease in proportion to the pulse intensity. However, there is no change in the transverse components of the average momentum, and the particle exits the pulse with $u_y = 1$ and $u_z = 1$. The amplitude of the longitudinal component of the oscillation also increases and decreases in proportion to the pulse intensity. However, because Eq. (14), which describes the relation between the longitudinal and transverse components of the momentum, is nonlinear, the longitudinal component of the average momentum changes. This change can be analyzed quantitatively. It follows from the t and x components of Eq. (28), and the assumed dependence of e on t–x, that

$$d_\tau\left(\upsilon_t - \upsilon_x\right) = 0. \tag{30}$$

Since υ_y and υ_z are constant, Eq. (29) reduces to

$$d_\tau\left[\left(\upsilon_t^2 - \upsilon_x^2\right)/2 - e^2/4\right] = 0. \tag{31}$$

By combining Eqs. (30) and (31) with the initial conditions, one can show that $\upsilon_t = 2 + e^2/4$ and $\upsilon_x = 1 + e^2/4$. At the peak of the pulse $\upsilon_x = 13/4$, in agreement with Fig. 69.25(a). Because the x component of the ponderomotive force is positive in the front of the pulse and negative in the back of the pulse, the guiding center is accelerated and decelerated by equal amounts. In this example the correspondence between the guiding-center motion and the particle motion is excellent.

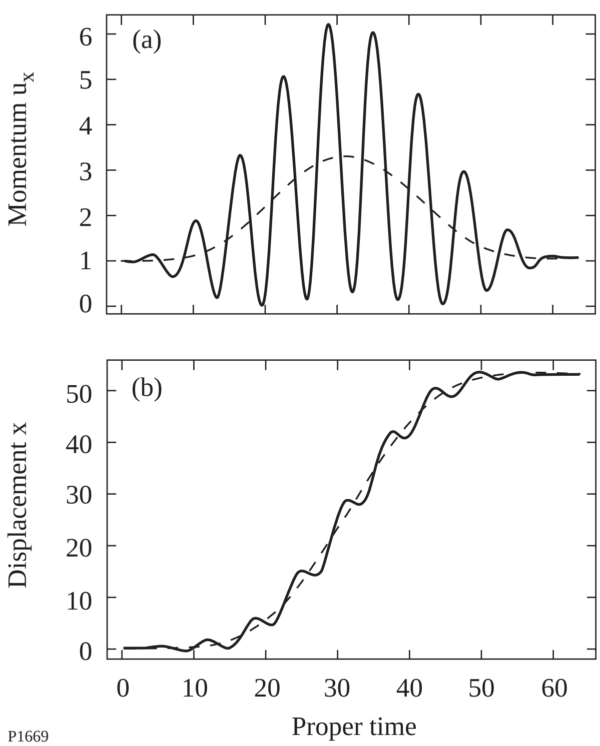

Figure 69.25
Particle motion (solid line) and guiding-center motion (dashed line) caused by a circularly polarized pulse with amplitude $e = 3\sin^2[0.05(t–x)]$. Initially, $u_x = 1$, $u_y = 1$, and $u_z = 1$. (a) The x component of the momentum. (b) The x component of the displacement caused by the pulse. The initial drift upon which this displacement is superimposed is not shown.

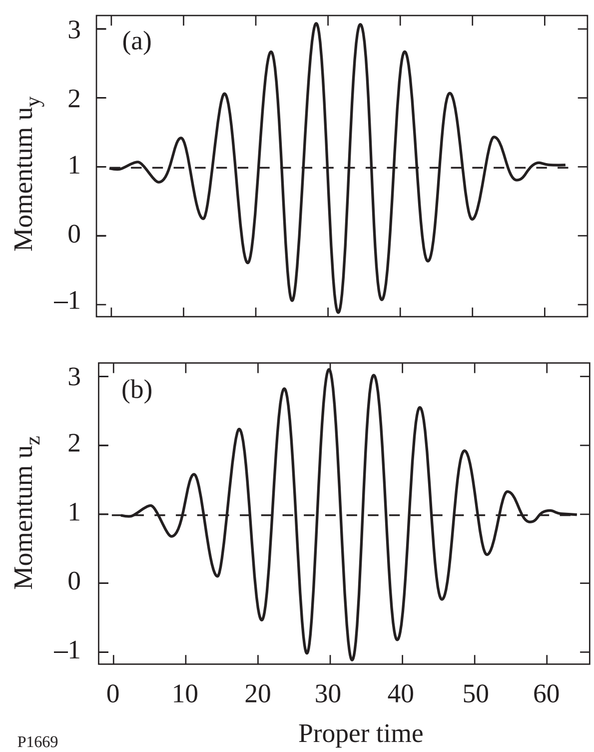

Figure 69.26
Particle motion (solid line) and guiding-center motion (dashed line) caused by a circularly polarized pulse with amplitude $e = 3\sin^2[0.05(t–x)]$. Initially, $u_x = 1$, $u_y = 1$, and $u_z = 1$. (a) The y component of the momentum. (b) The z component of the momentum.

The second example concerns a particle that is born inside a laser pulse by high-field ionization.[21] We considered a long pulse that is linearly polarized in the y direction, with $e = \cos^2$ (0.05 z), and chose $u_x(0) = 0$, $u_y(0) = 0$, and $u_z(0) = 0$. The resulting particle motion is illustrated in Figs. 69.27 and 69.28. The particle is born near the propagation axis of the pulse and is pushed outward by the z component of the ponderomotive force. As the particle moves outward, the amplitudes of the longitudinal and transverse components of the oscillation decrease in proportion to the pulse intensity. This transverse expulsion can be analyzed quantitatively. Since υ_t, υ_x, and υ_y are all constant, Eq. (29) reduces to

$$d_\tau\left(\upsilon_z^2/2 + e^2/4\right) = 0\,, \tag{32}$$

in which $\upsilon_z^2/2$ plays the role of kinetic energy and $e^2/4$ plays the role of potential energy. It follows from Eq. (32) and the initial conditions that $\upsilon_z^2 \approx \left(1 - e^2\right)/2$. As the guiding center exits the pulse, $\upsilon_z \approx 1/\sqrt{2}$, in agreement with Fig. 69.27(a). Although the particle is born at rest, it exits the pulse with $u_x \approx 3/4$ and $u_y \approx 1$. This behavior is consistent with Eqs. (16) and (19). In this example the correspondence between the guiding-center motion and the particle motion is excellent.

The third example concerns a particle that is injected into a laser pulse from the side. We considered a long pulse that is linearly polarized in the y direction, with $e = \sin^2(0.05y)$, and chose $u_x(0) = 0.0$, $u_y(0) = 0.7$, and $u_z(0) = 0.0$. The resulting particle motion is illustrated in Figs. 69.29 and 69.30. As the particle moves inward, the amplitudes of the longitudinal and transverse components of the oscillation increase in proportion to the pulse intensity. However, the y component of the ponderomotive force opposes the inward motion, and the particle is repelled just before it reaches the propagation axis of the pulse. As the particle moves outward, the amplitudes of the longitudinal and transverse components of the oscillation decrease in proportion to the pulse intensity. This transverse repulsion can be analyzed quantitatively. Since υ_t, υ_x, and υ_z are all constant, Eq. (29) reduces to

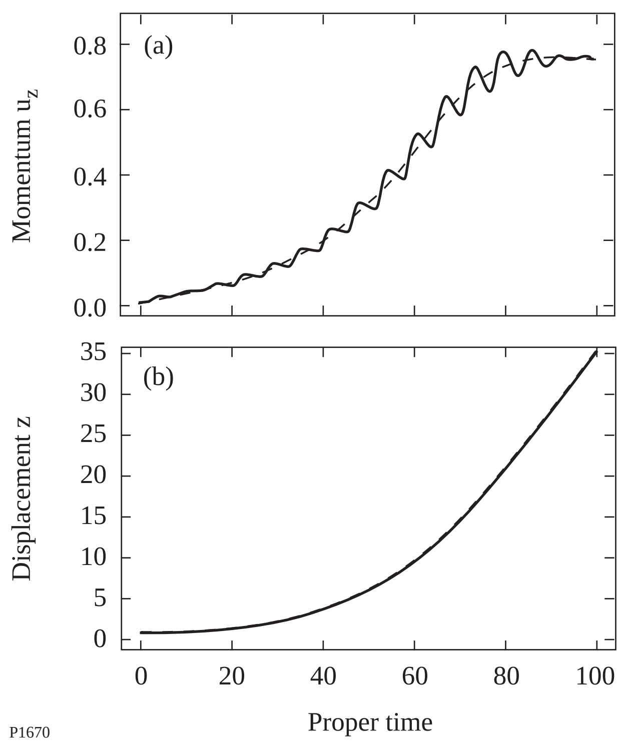

Figure 69.27
Particle motion (solid line) and guiding-center motion (dashed line) caused by a linearly polarized pulse with amplitude $e_y = \cos^2(0.05z)$. Initially, $u_x = 0$, $u_y = 0$, and $u_z = 0$. (a) The z component of the momentum. (b) The z component of the displacement.

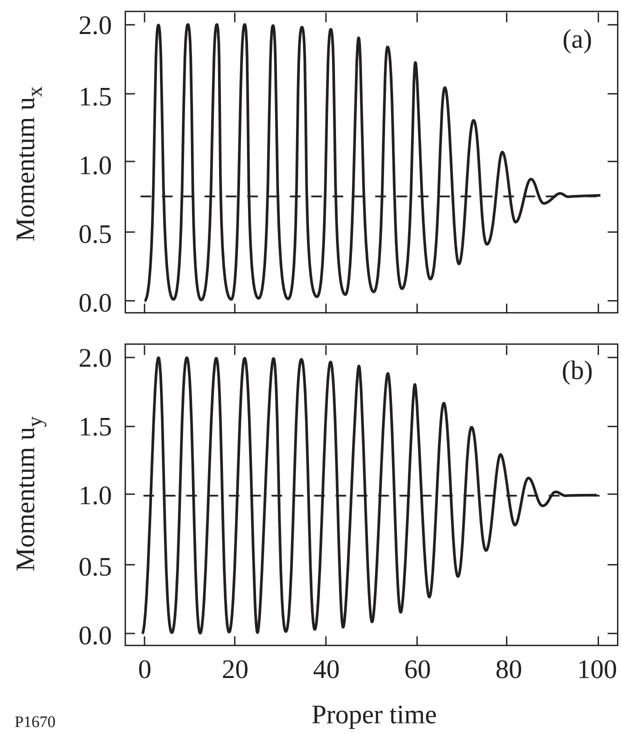

Figure 69.28
Particle motion (solid line) and guiding-center motion (dashed line) caused by a linearly polarized pulse with amplitude $e_y = \cos^2(0.05z)$. Initially, $u_x = 0$, $u_y = 0$, and $u_z = 0$. (a) The x component of the momentum. (b) The y component of the momentum.

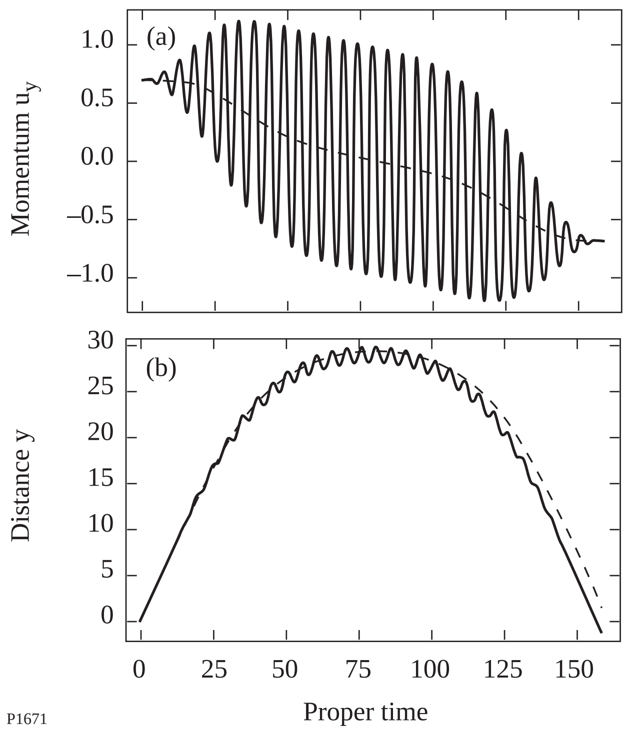

Figure 69.29
Particle motion (solid line) and guiding-center motion (dashed line) caused by a linearly polarized pulse with amplitude $e_y = \sin^2(0.05y)$. Initially, $u_x = 0.0$, $u_y = 0.7$, and $u_z = 0.0$. (a) The y component of the momentum. (b) The y component of the displacement.

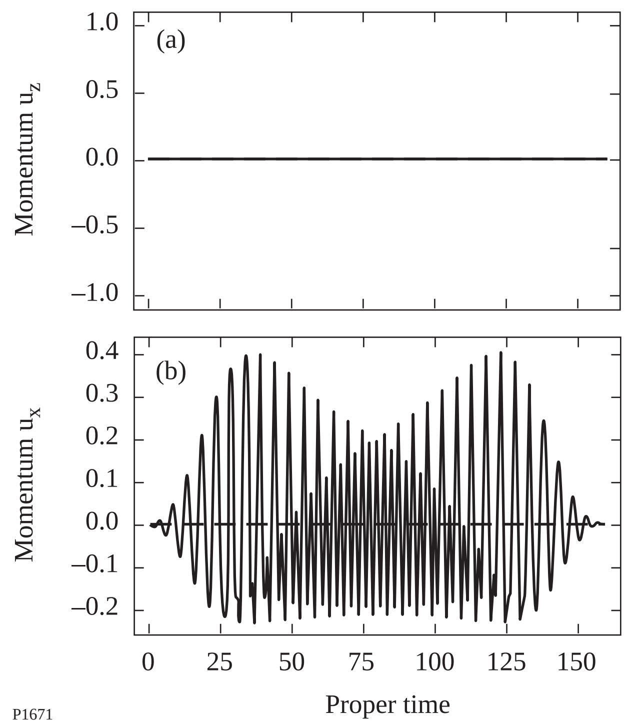

Figure 69.30
Particle motion (solid line) and guiding-center motion (dashed line) caused by a linearly polarized pulse with amplitude $e_y = \sin^2(0.05y)$. Initially, $u_x = 0.0$, $u_y = 0.7$, and $u_z = 0.0$. (a) The z component of the momentum. (b) The x component of the momentum.

$$d_\tau\left(\upsilon_y^2/2 + e^2/4\right) = 0 . \tag{33}$$

It follows from Eq. (33) and the initial conditions that $\upsilon_y^2 \approx \left(1 - e^2\right)/2$. The outward guiding-center trajectory is the inverse of the inward trajectory. In this example the correspondence between the guiding-center motion and the particle motion is good. We found the correspondence to be even better for gentler gradients in pulse intensity.

In Figs. 69.25–69.30 the particle and guiding-center positions were plotted as functions of the proper time. We verified numerically that plotting the spatial components of the guiding-center position as functions of the temporal component of the guiding-center position produces the correct guiding-center motion in the laboratory frame.

Multiple Scale Analysis of the Particle Motion

In this section we verify Eq. (28) analytically. Because the fast variation of the four-potential depends on the phase rather than the proper time, it is advantageous to change the independent variable in Eq. (8) from τ to ϕ. The result is

$$d_\phi\left(d_\tau\phi\, d_\phi x_\mu + a_\mu\right) = d_\phi x^\nu \partial_\mu a_\nu , \tag{34}$$

where $d_\tau\phi = \left(d_\phi x^\nu d_\phi x_\nu\right)^{-1/2}$. The resolution of Eq. (34) into longitudinal and transverse components is facilitated by the introduction of the four-vector k^μ, which is defined by the equation $\phi = k^\nu x_\nu$, and the four-vector l^μ, which is defined by the equations $l^\nu l_\nu = 0$, $k^\nu l_\nu = 2$, and $a^\nu l_\nu = 0$, where a^μ is the transverse four-potential of a plane wave. In the laboratory frame $k^\mu = (1,1,0,0)$ and $l^\mu = (1,–1,0,0)$. By using these four-vectors one can write

$$x^\mu = y^\mu + \theta k^\mu/2 + \phi l^\mu/2 , \tag{35}$$

where $\theta = l^\nu x_\nu$. The transverse position four-vector satisfies the equations $k^\nu y_\nu = 0$ and $l^\nu y_\nu = 0$. In a similar way, one can write

$$a^\mu = b^\mu + qk^\mu/2 + pl^\mu/2, \tag{36}$$

where the transverse four-potential satisfies the equations $k^\nu b_\nu = 0$ and $l^\nu b_\nu = 0$. By substituting the decompositions (35) and (36) into Eq. (34) and collecting like terms, one can show

$$\frac{d}{d\phi}\left(\frac{1}{\sigma}\frac{dy_\mu}{d\phi} + b_\mu\right) = \frac{\partial b^\nu}{\partial y^\mu}\frac{dy_\nu}{d\phi} + \frac{1}{2}\left(\frac{\partial p}{\partial y^\mu}\frac{d\theta}{d\phi} + \frac{\partial q}{\partial y^\mu}\right), \tag{37}$$

$$\frac{d}{d\phi}\left(\frac{1}{\sigma} + p\right) = 2\frac{\partial b^\nu}{\partial \theta}\frac{dy_\nu}{d\phi} + \frac{\partial p}{\partial \theta}\frac{d\theta}{d\phi} + \frac{\partial q}{\partial \theta}, \tag{38}$$

$$\frac{d}{d\phi}\left(\frac{1}{\sigma}\frac{d\theta}{d\phi} + q\right) = 2\frac{\partial b^\nu}{\partial \phi}\frac{dy_\nu}{d\phi} + \frac{\partial p}{\partial \phi}\frac{d\theta}{d\phi} + \frac{\partial q}{\partial \phi}, \tag{39}$$

where

$$\sigma = \left(d_\phi y^\nu d_\phi y_\nu + d_\phi \theta\right)^{1/2}. \tag{40}$$

Equation (39) can be derived from Eqs. (37) and (38), as shown in **Appendix B**, and need not be considered further.

One can solve Eqs. (37) and (38) by using multiple scale analysis. Let ε be a measure of the rate at which the wave amplitude varies relative to the rate at which the phase varies. We introduce the scales

$$\phi_0 = \phi, \quad \phi_1 = \varepsilon\phi \tag{41}$$

to resolve the fast oscillation and the slow change in the guiding-center drift, respectively. It follows that

$$\frac{d}{d\phi} = \frac{d}{d\phi_0} + \varepsilon\frac{d}{d\phi_1}. \tag{42}$$

We used the notation $d/d\phi_0$ and $d/d\phi_1$ in Eq. (42) to distinguish these convective derivatives from the partial derivatives of the four-potential. We assume that the dependent variables can be written as

$$\begin{aligned} y_\mu &\approx \varepsilon^{-1} y_\mu^{(-1)}(\phi_1) + y_\mu^{(0)}(\phi_0,\phi_1) + \varepsilon\, y_\mu^{(1)}(\phi_0,\phi_1), \\ \theta &\approx \varepsilon^{-1}\theta^{(-1)}(\phi_1) + \theta^{(0)}(\phi_0,\phi_1) + \varepsilon\,\theta^{(1)}(\phi_0,\phi_1). \end{aligned} \tag{43}$$

The variables $y_\mu^{(-1)}$ and $\theta^{(-1)}$ describe the guiding-center drift, which changes on the slow scale ϕ_1. The variables $y_\mu^{(0)}$ and $\theta^{(0)}$ describe the fast oscillation of the particle about the guiding center, the amplitude of which changes on the slow scale.

The four-potential satisfies Maxwell's wave equation[14]

$$\left(\partial^\lambda \partial_\lambda g_\nu^\mu - \partial^\mu \partial_\nu\right) a^\nu = 0, \tag{44}$$

where $g_\nu^\mu = \mathrm{diag}(1,-1,-1,-1)$ is the metric four-tensor. For a wave of constant amplitude, $a_\mu\left(x^\nu\right) = b_\mu^{(0)}\left(\phi_0\right)$. For a wave of variable amplitude we assume that

$$a_\mu\left(x^\nu\right) \approx a_\mu^{(0)}\left(\phi_0 \middle| \varepsilon x^\nu\right) + \varepsilon a_\mu^{(1)}\left(\phi_0 \middle| \varepsilon x^\nu\right). \tag{45}$$

Each contribution to the four-potential and its derivatives can be written approximately as

$$\begin{aligned} a\left(\phi_0 \middle|\ \varepsilon y_\nu, \varepsilon\theta, \varepsilon\phi\right) \approx\ & a\left(\phi_0 \middle| y_\nu^{(-1)}, \theta^{(-1)}, \phi_1\right) \\ & + \varepsilon y^{(0)\nu}\partial_\nu a\left(\phi_0 \middle| y_\nu^{(-1)}, \theta^{(-1)}, \phi_1\right) \\ & + \varepsilon\theta^{(0)}\partial_\theta a\left(\phi_0 \middle| y_\nu^{(-1)}, \theta^{(-1)}, \phi_1\right). \end{aligned} \tag{46}$$

The first term on the right side of Eq. (46) is the contribution evaluated at the guiding center, and the second and third terms are the deviations from this average contribution that are felt by the particle as it oscillates about the guiding center. The corresponding approximation for the convective derivative of the four-potential is discussed in **Appendix C**. Henceforth, we will use $\bar{a}$ to denote the guiding-center contribution

$$a\left(\phi_0 \middle| y_\nu^{(-1)}, \theta^{(-1)}, \phi_1\right).$$

To proceed further one substitutes Eqs. (42), (43), and (46) in Eqs. (37) and (38) and collects terms of like order. The order ε^{-1} equations are satisfied identically by Ansaetze (43).

The order 1 equations are

$$\frac{d}{d\phi_0}\left[\frac{1}{\sigma^{(0)}}\frac{dy_\mu^{(0)}}{d\phi_0} + \bar{b}_\mu^{(0)}\right] = 0, \tag{47}$$

$$\frac{d}{d\phi_0}\left[\frac{1}{\sigma^{(0)}}\right]=0\,, \tag{48}$$

where

$$\sigma^{(0)}=\Big\{\Big[d_0 y^{(0)\nu}+d_1 y^{(-1)\nu}\Big] \times\Big[d_0 y_\nu^{(0)}+d_1 y_\nu^{(-1)}\Big]+d_0\theta^{(0)}+d_1\theta^{(-1)}\Big\}^{1/2} \tag{49}$$

and $d_n = d/d\phi_n$.

Equation (47) is the analog of Eq. (10). It follows from the former equation that

$$d_0 y_\mu^{(0)} = -\sigma^{(0)}\,\bar{b}_\mu^{(0)}\,. \tag{50}$$

The arbitrary function of ϕ_1 that results from the ϕ_0 integration can be neglected because $y_\mu^{(-1)}$ already accounts for the slowly varying drift with which this function is associated. Equations (48) and (49) do not resemble any of the equations in the section **Particle Motion in a Plane Wave**. However, different forms of the latter equations are discussed in **Appendix A**, from which it is clear that Eqs. (48) and (49) comprise the analog of Eq. (A9). It follows from Eq. (48) that $\sigma^{(0)}$ is a function of ϕ_1 alone. This result is the analog of Eq. (15) and facilitates the integration of Eq. (50). By combining Eqs. (49) and (50), and equating the oscillatory and slowly varying terms that result, one can show that

$$d_0\theta^{(0)}=2\sigma^{(0)}d_1 y^{(-1)\nu}\,\bar{b}_\nu^{(0)} +\left[\sigma^{(0)}\right]^2\left[\left\langle\bar{b}^{(0)\nu}\,\bar{b}_\nu^{(0)}\right\rangle-\bar{b}^{(0)\nu}\,\bar{b}_\nu^{(0)}\right] \tag{51}$$

and

$$d_1\theta^{(-1)}+d_1 y^{(-1)\nu}d_1 y_\nu^{(-1)} =\left[\sigma^{(0)}\right]^2\left[1-\left\langle\bar{b}^{(0)\nu}\,\bar{b}_\nu^{(0)}\right\rangle\right]. \tag{52}$$

Equation (51) is the analog of Eq. (17) and the oscillatory part of Eq. (A7), and is easy to integrate.

Now consider the initial condition on the order-1 four-momentum. Consistent with Eq. (35), one can write the initial four-momentum as

$$u^\mu(0)=v^\mu(0)+l^\nu u_\nu(0)k^\mu/2+k^\nu u_\nu(0)l^\mu/2\,. \tag{53}$$

It follows immediately that

$$d_1 y_\mu^{(-1)}(0)=\sigma^{(0)}v_\mu(0)-d_0 y_\mu^{(0)}(0)\,, \tag{54}$$

$$d_1\theta^{(-1)}(0)=\sigma^{(0)}l^\nu u_\nu(0)-d_0\theta^{(0)}(0)\,. \tag{55}$$

Equation (54) is the analog of Eqs. (16), and Eq. (55) is consistent with Eqs. (17) and (18).

The order ε equations are

$$\begin{aligned}
&\frac{d}{d\phi_1}\left\{\frac{1}{\sigma^{(0)}}\left[\frac{dy_\mu^{(0)}}{d\phi_0}+\frac{dy_\mu^{(-1)}}{d\phi_1}\right]+\bar{b}_\mu^{(0)}\right\}\\
&+\quad\frac{d}{d\phi_0}\left\{\frac{1}{\sigma^{(0)}}\left[\frac{dy_\mu^{(1)}}{d\phi_0}+\frac{dy_\mu^{(0)}}{d\phi_1}\right]+\bar{b}_\mu^{(1)}\right\}\\
&-\quad\frac{d}{d\phi_0}\left\{\frac{\sigma^{(1)}}{2\left[\sigma^{(0)}\right]^3}\left[\frac{dy_\mu^{(0)}}{d\phi_0}+\frac{dy_\mu^{(-1)}}{d\phi_1}\right]\right\}\\
&=\quad\left[\frac{dy^{(0)\nu}}{d\phi_0}+\frac{dy^{(-1)\nu}}{d\phi_1}\right]\frac{\partial\bar{b}_\nu^{(0)}}{\partial y^\mu}
\end{aligned} \tag{56}$$

and

$$\begin{aligned}
&\frac{d}{d\phi_1}\left[\frac{1}{\sigma^{(0)}}\right]-\frac{d}{d\phi_0}\left\{\frac{\sigma^{(1)}}{2\left[\sigma^{(0)}\right]^3}+\bar{p}^{(1)}\right\}\\
&=\;2\left[\frac{dy^{(0)\nu}}{d\phi_0}+\frac{dy^{(-1)\nu}}{d\phi_1}\right]\frac{\partial\bar{b}_\nu^{(0)}}{\partial\theta}\,,
\end{aligned} \tag{57}$$

where

$$\sigma^{(1)} = \frac{d\theta^{(1)}}{d\phi_0} + \frac{d\theta^{(0)}}{d\phi_1}$$

$$+ 2\left[\frac{dy^{(0)\nu}}{d\phi_0} + \frac{dy^{(-1)\nu}}{d\phi_1}\right]\left[\frac{dy^{(1)\nu}}{d\phi_0} + \frac{dy^{(0)\nu}}{d\phi_1}\right], \qquad (58)$$

and $\bar{b}_\mu^{(1)}$ and $\bar{p}^{(1)}$ represent the sum of the order ε four-potential and the order ε corrections to the order-1 four-potential caused by the oscillation of the particle about the guiding center.

Although Eqs. (56)–(58) are lengthy, they do not need to be solved in their entirety. By equating the slowly varying terms in Eqs. (56) and (57), one can show that

$$\frac{1}{\sigma^{(0)}}\frac{d}{d\phi_1}\left[\frac{1}{\sigma^{(0)}}\frac{dy_\mu^{(-1)}}{d\phi_1}\right] = -\frac{1}{2}\frac{\partial\left\langle \bar{b}^{(0)\nu}\,\bar{b}_\nu^{(0)}\right\rangle}{\partial y^\mu} \qquad (59)$$

and

$$\frac{1}{\sigma^{(0)}}\frac{d}{d\phi_1}\left[\frac{1}{\sigma^{(0)}}\right] = -\frac{\partial\left\langle \bar{b}^{(0)\nu}\,\bar{b}_\nu^{(0)}\right\rangle}{\partial\theta}. \qquad (60)$$

It follows from Eq. (52) that

$$\frac{1}{\sigma^{(0)}}\frac{d}{d\phi_1}\left[\frac{1}{\sigma^{(0)}}\frac{d\theta^{(-1)}}{d\phi_1}\right]$$

$$= \frac{1-\left\langle \bar{b}^{(0)\nu}\,\bar{b}_\nu^{(0)}\right\rangle}{\sigma^{(0)}}\frac{d\sigma^{(0)}}{d\phi_1} - \frac{d\left\langle \bar{b}^{(0)\nu}\,\bar{b}_\nu^{(0)}\right\rangle}{d\phi_1}$$

$$-\frac{1}{\sigma^{(0)}}\frac{d}{d\phi_1}\left\{\sigma^{(0)}\left[\frac{1}{\sigma^{(0)}}\frac{dy^{(-1)\nu}}{d\phi_1}\right]\left[\frac{1}{\sigma^{(0)}}\frac{dy_\nu^{(-1)}}{d\phi_1}\right]\right\}. \qquad (61)$$

When applied to any guiding-center quantity, the operator

$$\frac{d}{d\phi_1} = \frac{dy^{(-1)\nu}}{d\phi_1}\frac{\partial}{\partial y^\nu} + \frac{\partial}{\partial\phi_1} + \frac{d\theta^{(-1)}}{d\phi_1}\frac{\partial}{\partial\theta}. \qquad (62)$$

By combining Eq. (61) with Eqs. (59), (60), and (62), one can show that

$$\frac{1}{\sigma^{(0)}}\frac{d}{d\phi_1}\left[\frac{1}{\sigma^{(0)}}\frac{d\theta^{(-1)}}{d\phi_1}\right] = -\frac{\partial\left\langle \bar{b}^{(0)\nu}\,\bar{b}_\nu^{(0)}\right\rangle}{\partial\phi_1}. \qquad (63)$$

Recall that the preceding derivation of Eq. (63) is based on Eq. (38). Had we analyzed Eq. (39) instead, we would have needed to determine $b_\nu^{(1)}$, $p^{(1)}$, $q^{(1)}$, and $y_\nu^{(1)}$ explicitly.

In the notation of this section, Eq. (28) can be rewritten as

$$\frac{d^2 x_\mu^{(-1)}}{d\tau_1^2} = -\frac{1}{2}\frac{\partial\left\langle \bar{b}^{(0)\nu}\,\bar{b}_\nu^{(0)}\right\rangle}{\partial x_1^\mu}, \qquad (64)$$

where $\tau_1 = \varepsilon\tau$ and $x_1^\mu = \varepsilon x^\mu$. Since $d\phi_1/d\tau_1 \approx 1/\sigma^{(0)}$, Eq. (59) is the transverse part of Eq. (64). By contracting Eq. (64) with k^μ and l^μ, and using the identities $k^\mu\partial_\mu = 2\partial_\theta$ and $l^\mu\partial_\mu = 2\partial_\phi$, and the fact that $\phi \approx k^\mu x_\mu^{(-1)}$, one can show that Eqs. (60) and (63) are equivalent to the longitudinal part of Eq. (64). Thus, Eq. (28) is correct.

Finally, notice that Eq. (64) for the guiding-center drift is written in terms of the proper time, which includes the effects of the oscillation about the guiding center. Although this fact does not affect the utility of Eq. (64), it calls into question the aesthetic qualities of the equation. Just as the proper time is defined by the equation $d\tau = \left(dx^\nu dx_\nu\right)^{1/2}$, one can define the drift time by the equation $ds = \left[dx^{(-1)\nu}\,dx_\nu^{(-1)}\right]^{1/2}$.

It follows from this definition, Eq. (52), and the discussion of the preceding paragraph that

$$\frac{ds_1}{d\tau_1} = \left[1-\left\langle \bar{b}^{(0)\nu}\,\bar{b}_\nu^{(0)}\right\rangle\right]^{1/2}. \qquad (65)$$

Equation (65) can be used to write Eq. (28) in terms of the drift time.

Summary

In this article we solved the equation of motion for an electron in a plane wave. We used this solution and the principle of Lorentz covariance to deduce a formula for the ponderomotive force exerted by an intense laser pulse on an electron. We verified this formula numerically, for three cases of current interest, and analytically, using the method of multiple scales.

The aforementioned formula can be used to study the effects of the radial ponderomotive force on laser-plasma interactions. For particle accelerators, these effects include the divergence of an electron bunch that is accelerated by a laser pulse,[22] the relativistic focusing of the pulse, and electron cavitation and magnetic field generation in the wake of the pulse.

ACKNOWLEDGMENT

This work was supported by the National Science Foundation under Contracts No. PHY-9057093 and PHY-9415583, the U.S. Department of Energy Office of Inertial Confinement Fusion under Cooperative Agreement No. DE-FC03-92SF19460, the University of Rochester, and the New York State Energy Research and Development Authority. The support of DOE does not constitute an endorsement by DOE of the views expressed in this article.

Appendix A: Covariant Analysis of the Particle Motion in a Plane Wave

The motion of a charged particle in an electromagnetic field is governed by Eq. (8). For a plane wave the four-potential a^μ is a function of the phase $\phi = k^\nu x_\nu$. It follows that $\partial_\mu a_\nu = k_\mu a'_\nu$, where $' = d/d\phi$, and, hence, that

$$d_\tau\left(u_\mu + a_\mu\right) = u^\nu k_\mu a'_\nu\,. \tag{A1}$$

By substituting the decomposition

$$u_\mu(\tau) = v_\mu(\tau) + k^\nu u_\nu(\tau) l_\mu/2 + l^\nu u_\nu(\tau) k_\mu/2 \tag{A2}$$

into Eq. (A1), where l^ν was defined after Eq. (8), and v_μ satisfies the equations $k^\nu v_\nu = 0$ and $l^\nu v_\nu = 0$, one can show that

$$d_\tau\left(v_\mu + a_\mu\right) = 0\,, \quad d_\tau\left(k^\nu u_\nu\right) = 0\,, \quad d_\tau\left(l^\nu u_\nu\right) = 2v^\nu a'_\nu\,. \tag{A3}$$

It follows from the first of Eqs. (A3) that

$$v_\mu(\tau) = v_\mu(0) + a_\mu(0) - a_\mu(\tau)\,. \tag{A4}$$

It follows from the second of Eqs. (A3) that

$$k^\nu u_\nu(\tau) = k^\nu u_\nu(0) \tag{A5}$$

and, hence, that

$$\phi = k^\nu u_\nu(0)\tau\,. \tag{A6}$$

Equations (A4) and (A6) determine $v_\mu(\tau)$ explicitly. There are at least three ways to obtain an expression for $l^\nu u_\nu$. In the first approach, one uses Eq. (A4) to rewrite the right side of the third of Eqs. (A3) in terms of a^μ. It follows from this equation and Eq. (A6) that

$$\begin{aligned} l^\nu u_\nu(\tau) = {} & l^\nu u_\nu(0) + 2\left[v^\nu(0) + a^\nu(0)\right] \\ & \times\left[a_\nu(\tau) - a_\nu(0)\right]/k^\nu u_\nu(0) \\ & + \left[a^\nu(0)a_\nu(0) - a^\nu(\tau)a_\nu(\tau)\right]/k^\nu u_\nu(0)\,. \end{aligned} \tag{A7}$$

In the second approach, one uses Eq. (A4) to rewrite the right side of the third of Eqs. (A3) in terms of v^μ. It follows from this equation and Eq. (A6) that

$$\begin{aligned} l^\nu u_\nu(\tau) = {} & l^\nu u_\nu(0) + \left[v^\nu(0)v_\nu(0)\right. \\ & \left. - v^\nu(\tau)v_\nu(\tau)\right]/k^\nu u_\nu(0)\,. \end{aligned} \tag{A8}$$

In the third approach one uses decomposition (A2) to rewrite the identity $u^\nu u_\nu = 1$ as

$$\left(k^\nu u_\nu\right)\left(l^\nu u_\nu\right) + v^\nu v_\nu = 1\,. \tag{A9}$$

Since $k^\nu u_\nu$ and $v^\nu v_\nu$ are known quantities, Eq. (A9) provides a third expression for $l^\nu u_\nu$. By rewriting the 1 on the right side of Eq. (A9) in terms of the initial values of the quantities on the left side, one can rewrite Eq. (A9) in the form of Eq. (A8). All three approaches have their uses. Equation (A4) is the covariant version of Eq. (11), and Eqs. (A5) and (A8) are the covariant versions of Eq. (14) for $u_\parallel$ and its analog for γ.

Appendix B: Covariant Lagrangian for the Particle Motion

For a particle in an electromagnetic field the normalized motion[19]

$$S = -\int\left[\left(dx^{\nu}dx_{\nu}\right)^{1/2} + a^{\nu}dx_{\nu}\right]. \tag{B1}$$

Traditionally, one parameterizes the particle motion in terms of the proper time τ, which is a Lorentz invariant. In this case

$$S = -\int\left[\left(d_{\tau}x^{\nu}d_{\tau}x_{\nu}\right)^{1/2} + a^{\nu}d_{\tau}x_{\nu}\right]d\tau. \tag{B2}$$

By applying the Euler-Lagrange equations to the integrand of Eq. (B2), one finds that

$$d_{\tau}\left(d_{\tau}x_{\mu} + a_{\mu}\right) = d_{\tau}x^{\nu}\partial_{\mu}a_{\nu}, \tag{B3}$$

in agreement with Eq. (8). Alternately, one can parameterize the particle motion by the phase $\phi = k^{\nu}x_{\nu}$, which is also a Lorentz invariant. In this case

$$S = -\int\left[\left(d_{\phi}x^{\nu}d_{\phi}x_{\nu}\right)^{1/2} + a^{\nu}d_{\phi}x_{\nu}\right]d\phi. \tag{B4}$$

By using the decompositions (35) and (36) one can rewrite Eq. (B4) as

$$\begin{aligned} S &= -\int\Big[\left(d_{\phi}y^{\nu}d_{\phi}y_{\nu} + d_{\phi}\theta\right)^{1/2} \\ &\qquad + b^{\nu}d_{\phi}y_{\nu} + p d_{\phi}\theta/2 + q/2\Big]d\phi. \end{aligned} \tag{B5}$$

By applying the Euler-Lagrange equations to the integrand of Eq. (B5), one can show that

$$\begin{aligned} &\frac{d}{d\phi}\left[\frac{1}{\left(d_{\phi}y^{\nu}d_{\phi}y_{\nu} + d_{\phi}\theta\right)^{1/2}}\frac{dy_{\mu}}{d\phi} + b_{\mu}\right] \\ &= \frac{\partial b^{\nu}}{\partial y^{\mu}}\frac{dy_{\nu}}{d\phi} + \frac{1}{2}\left(\frac{\partial p}{\partial y^{\mu}}\frac{d\theta}{d\phi} + \frac{\partial q}{\partial y^{\mu}}\right), \end{aligned} \tag{B6}$$

$$\begin{aligned} &\frac{d}{d\phi}\left[\frac{1}{\left(d_{\phi}y^{\nu}d_{\phi}y_{\nu} + d_{\phi}\theta\right)^{1/2}} + q\right] \\ &= 2\frac{\partial b^{\nu}}{\partial\theta}\frac{dy_{\nu}}{d\phi} + \frac{\partial p}{\partial\theta}\frac{d\theta}{d\phi} + \frac{\partial q}{\partial\theta}, \end{aligned} \tag{B7}$$

in agreement with Eqs. (37) and (38). One can reproduce Eq. (39) by multiplying Eq. (B6) by $-2d_{\phi}y^{\mu}$ and Eq. (B7) by $-d_{\phi}\theta$, and adding the resulting equations.

Appendix C: Evaluation of the Four-Potential

The left side of Eq. (34) contains the term $da_{\mu}/d\phi$, which must be evaluated at the position of the particle. In the section **Multiple Scale Analysis of the Particle Motion** we used Eqs. (42), (43), and (46) to make a guiding-center expansion of a_{μ} *before* we took the convective derivative. Specifically, we wrote

$$d_{\phi}a_{\mu} \approx \left[d_0 + \varepsilon\, d_1\right]\left[a_{\mu}^{(0)} + \varepsilon\, a^{(1)}\right], \tag{C1}$$

where

$$a_{\mu}^{(0)} = \bar{a}_{\mu} \tag{C2}$$

is the four-potential evaluated at the guiding center and

$$\bar{a}_{\mu}^{(1)} = y^{(0)\nu}\partial_{\nu}\bar{a}_{\mu} + \theta^{(0)}\partial_{\theta}\bar{a}_{\mu} \tag{C3}$$

is the correction to the four-potential caused by the oscillation of the particle about the guiding center. Since the guiding-center coordinates $y^{(-1)}$ and $\theta^{(-1)}$ are functions of ϕ_1 by construction,

$$\frac{d\bar{a}_{\mu}}{d\phi_0} = \frac{\partial\bar{a}_{\mu}}{\partial\phi_0} \tag{C4}$$

and

$$\frac{d\bar{a}_{\mu}}{d\phi_1} = \frac{dy^{(-1)\nu}}{d\phi_1}\frac{\partial\bar{a}_{\mu}}{\partial y^{\nu}} + \frac{\partial\bar{a}_{\mu}}{\partial\phi_1} + \frac{d\theta^{(-1)}}{d\phi_1}\frac{\partial\bar{a}_{\mu}}{\partial\theta}. \tag{C5}$$

It follows from Eqs. (C1), (C4), and (C5) that

$$\left[\frac{da_\mu}{d\phi}\right]^{(0)} = \frac{\partial \bar{a}_\mu}{\partial \phi_0} \tag{C6}$$

and

$$\begin{aligned}\left[\frac{da_\mu}{d\phi}\right]^{(1)} &= \frac{dy^{(-1)\nu}}{d\phi_1}\frac{\partial \bar{a}_\mu}{\partial y^\nu} + \frac{\partial \bar{a}_\mu}{\partial \phi_1} + \frac{d\theta^{(-1)}}{d\phi_1}\frac{\partial \bar{a}_\mu}{\partial \theta} \\ &+ \frac{dy^{(0)\nu}}{d\phi_0}\frac{\partial \bar{a}_\mu}{\partial y^\nu} + y^{(0)\nu}\frac{\partial^2 \bar{a}_\mu}{\partial \phi_0 \partial y^\nu} \\ &+ \frac{d\theta^{(0)}}{d\phi_0}\frac{\partial \bar{a}_\mu}{\partial \theta} + \theta^{(0)}\frac{\partial^2 \bar{a}_\mu}{\partial \phi_0 \partial \theta}.\end{aligned} \tag{C7}$$

Alternately, one can write

$$\frac{da_\mu}{d\phi} = \frac{dy^\nu}{d\phi}\frac{\partial a_\mu}{\partial y^\nu} + \frac{\partial a_\mu}{\partial \phi} + \frac{d\theta}{d\phi}\frac{\partial a_\mu}{\partial \theta}, \tag{C8}$$

in which the guiding-center expansion is made *after* the partial derivatives are taken. Since the variation of a_μ with the position variables y^ν and θ is slow,

$$\begin{aligned}\frac{da_\mu}{d\phi} &\approx \varepsilon\left[\frac{dy^{(0)\nu}}{d\phi_0} + \frac{dy^{(-1)\nu}}{d\phi_1}\right]\frac{\partial a_\mu}{\partial y^\nu} + \frac{\partial a_\mu}{\partial \phi_0} \\ &+ \varepsilon\frac{\partial a_\mu}{\partial \phi_1} + \varepsilon\left[\frac{d\theta^{(0)}}{d\phi_0} + \frac{d\theta^{(-1)}}{d\phi_1}\right]\frac{\partial a_\mu}{\partial \theta}.\end{aligned} \tag{C9}$$

The derivatives of the four-potentials appearing in the order ε terms can be approximated by their guiding-center values. The remaining term

$$\frac{\partial a_\mu}{\partial \phi_0} \approx \frac{\partial \bar{a}_\mu}{\partial \phi_0} + \varepsilon y^{(0)\nu}\frac{\partial^2 \bar{a}_\mu}{\partial y^\nu \partial \phi_0} + \varepsilon\theta^{(0)}\frac{\partial^2 \bar{a}_\mu}{\partial \theta \partial \phi_0}. \tag{C10}$$

Equations (C9) and (C10) are equivalent to Eqs. (C4) and (C5). This result shows that the guiding-center expansion discussed previously was made consistently. The expansion based on Eq. (C1) is better because it facilitates the identification of combinations of terms that are oscillatory and, hence, do not affect the guiding-center motion.

REFERENCES

1. H. A. H. Boot and R. B. R. S. Harvie, Nature **180**, 1187 (1957).
2. H. A. H. Boot, S. A. Self, and R. B. R. S. Harvie, J. Electron. Control **IV**, 434 (1958).
3. A. V. Gaponov and M. A. Miller, Sov. Phys.-JETP **7**, 168 (1958).
4. A. V. Gaponov and M. A. Miller, Sov. Phys.-JETP **7**, 515 (1958).
5. G. A. Askar'yan, Sov. Phys.-JETP **15**, 1088 (1962).
6. N. J. Phillips and J. J. Sanderson, Phys. Lett. **21**, 533 (1966).
7. T. W. B. Kibble, Phys. Rev. Lett. **16**, 1054 (1966).
8. T. W. B. Kibble, Phys. Rev. **150**, 1060 (1966).
9. F. F. Chen, *Introduction to Plasma Physics and Controlled Fusion*, 2nd ed. (Plenum Press, New York, 1984), Vol. 1, p. 305.
10. W. L. Kruer, *The Physics of Laser Plasma Interactions*, Frontiers in Physics, Vol. 73 (Addison-Wesley, Redwood City, CA, 1988), p. 60.
11. R. S. Craxton, R. L. McCrory, and J. M. Soures, Sci. Am. **255**, 68 (1986).
12. J. M. Dawson, Sci. Am. **260**, 54 (1989).
13. M. D. Perry and G. Mourou, Science **264**, 917 (1994).
14. C. J. McKinstrie and D. F. DuBois, Phys. Fluids **31**, 278 (1988).
15. H. Goldstein, *Classical Mechanics*, 2nd ed. (Addison-Wesley, Reading, MA, 1980), p. 303.
16. D. M. Volkov, Z. Phys. **94**, 250 (1935).
17. T. J. M. Boyd and J. J. Sanderson, *Plasma Dynamics* (Barnes and Noble, New York, 1969), p. 26.
18. P. C. Clemmow and J. P. Dougherty, *Electrodynamics of Particles and Plasmas* (Addison-Wesley, Reading, MA, 1969), p. 122.
19. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Pergamon Press, Oxford, 1975), pp. 45, 112, and 118.
20. E. A. Startsev and C. J. McKinstrie, Bull. Am. Phys. Soc. **40**, 1723 (1995).
21. C. I. Moore, J. P. Knauer, and D. D. Meyerhofer, Phys. Rev. Lett. **74**, 2439 (1995).
22. C. J. McKinstrie and E. A. Startsev, Phys. Rev. E **54**, R1070 (1996).