Transport and Sound Waves in Plasmas with Light and Heavy Ions

Ion-transport coefficients are important in various aspects of plasma physics. Some of the most commonly used transport formulas have been derived by Braginskii.[1] They are obtained by assuming a fully ionized, single-ion-species plasma. Typical laboratory plasmas, however, may involve more than one species of ions. For example, in magnetic-fusion devices, high-Z impurities can be present within the DT fuel. In inertial-confinement fusion plastic materials are commonly used as ablators, which then give rise to carbon and hydrogen ions. To calculate the transport of ions in such plasmas, within the framework of single-fluid theory, it is usual to employ an average-ion model in conjunction with the Braginskii formulas. The aim of this article is to show that such a simple procedure can considerably underestimate the levels of thermal diffusion; viscosity, and joule heating for a mixture of light and heavy ions. Implications for the damping of ion-acoustic waves will be shown.

We start by recalling the formulas for the ion thermal conductivity and viscosity coefficients in an unmagnetized single-ion-species plasma. They are, respectively (in cgs units),[1]

$$\kappa_i = \gamma_i \frac{n_i T_i \tau_i}{m_i} \tag{1}$$

and

$$\eta_i = \mu_i n_i T_i \tau_i \,, \tag{2}$$

where $\gamma_i = 3.91$, $\mu_i = 0.96$, n_i is the number density, m_i is the mass, and T_i is the temperature (in ergs). The ion-ion collision time for 90° angular scattering is given by

$$\tau_i = \frac{3\sqrt{m_i}\, T_i^{3/2}}{4\sqrt{\pi} e^4 n_i \left(Z_i^2\right)^2 \ln \Lambda_i} \,, \tag{3}$$

where e is the magnitude of the electron charge and $\ln\Lambda_i$ is the Coulomb logarithm.

Let us now consider a fully ionized plasma composed of approximately equal numbers of light and heavy ions (identified by l and h, respectively). It is clear from the above equations that if $Z_h >> Z_l$, the transport will be dominated by the light species. However, since it is convenient to use a single-ion-species formalism, rather than treat the transport of each species separately, it is common to use Eqs. (1)–(3) with some appropriate average (denoted here by $\langle\,\rangle$) for m_i and Z_i. A popular approach is to use

$$\langle m_i \rangle = \left(n_l m_l + n_h m_h\right)/n_i \,,$$

$$\langle Z_i \rangle = \left(n_l Z_l + n_h Z_h\right)/n_i \,,$$

and

$$\langle Z_i^2 \rangle = \left(n_l Z_l^2 + n_h Z_h^2\right)/n_i \,,$$

where

$$n_i = n_l + n_h \,.$$

To investigate the accuracy of this averaging procedure we need to recalculate the transport coefficients. The linearized Fokker-Planck equation, written in the frame of the light-ion species with mean velocity $\mathbf{u}_l$ (obtained by expanding the distribution function as $f = f_0 + \mathbf{w}\cdot\mathbf{f}_1/w$, where $\mathbf{w} = \mathbf{v} - \mathbf{u}_l$ is the intrinsic velocity), is given by[2]

$$\left(\mathbf{C}_1^{ll} + \mathbf{C}_1^{lh}\right) = w\nabla f_0^l + \left(\frac{Z_l e}{m_l}\mathbf{E} - \frac{d\mathbf{u}_l}{dt}\right)\frac{\partial f_0^l}{\partial w} \,, \tag{4}$$

where $d/dt = \partial/\partial t + \mathbf{u}_l \cdot \nabla$, f_0 is the isotropic Maxwellian distribution, $\mathbf{f}_1$ is the anisotropic part of the distribution (responsible for the transport), $\mathbf{E}$ is the electric field, and $\mathbf{C}_1{}^{ll}$ and $\mathbf{C}_1{}^{lh}$ are the anisotropic parts of the collision operators acting on $\mathbf{f}_1$. Equation (4) has been derived with the standard assumptions of strong collisionality (which imply that $|\mathbf{f}_1| << f_0$) and negligible contribution from electron momentum exchange. Indeed, in the absence of l-h collisions, Eq. (4) predicts the classical single-ion-species thermal conductivity of Eq. (1). In our case, however, $\mathbf{C}_1^{lh}/\mathbf{C}_1^{ll} \sim n_h Z_h^2 / n_l Z_l^2 >> 1$ means that l-h collisions dominate over l-l collisions. Furthermore, the collision operator $\mathbf{C}_1^{lh}$ may be considerably simplified in the limit $m_h >> m_l$ to become[2]

$$\mathbf{C}_1^{lh} \approx -\frac{n_h Y_{lh}}{w^3}\left[\mathbf{f}_1^l + \left(\mathbf{u}_h - \mathbf{u}_l\right)\frac{\partial f_0^l}{\partial w}\right],$$

where

$$Y_{lh} = \frac{4\pi Z_l^2 Z_h^2 e^4 \ln \Lambda_{lh}}{m_l^2}$$

and $\mathbf{u}_h$ is the mean velocity of the h species (necessary to ensure momentum conservation).

Substituting this simplified collision operator back into Eq. (4) and expanding the right-hand side of that equation yields

$$\mathbf{f}_1^l = -\frac{w^4}{n_h Y_{lh}}\left[\left(\frac{m_l w^2}{2T_l} - \frac{5}{2}\right)\frac{\nabla T_l}{T_l}\right.$$

$$\left. + \frac{1}{p_l}\left(\nabla p_l - Z_l e n_l \mathbf{E} + n_l m_l \frac{d\mathbf{u}_l}{dt}\right)\right] f_0^l - \left(\mathbf{u}_h - \mathbf{u}_l\right)\frac{\partial f_0^l}{\partial w}. \quad (5)$$

We note that this equation is equivalent to the one used for modeling electron transport in high-Z plasmas.

Substituting Eq. (5) into the heat flow formula,

$$\mathbf{q}_l = \frac{2\pi}{3} m_l \int_0^\infty dw w^5 \mathbf{f}_1^l \,, \quad (6)$$

and using the velocity moment $\int dw w^3 \mathbf{f}_1^l$ to substitute for $\left(\nabla p_l - Z_l e n_l \mathbf{E} + n_l m_l d\mathbf{u}_l/dt\right)$, we obtain

$$\mathbf{q}_l = -\kappa_{lh}\nabla T_l + \beta_0 n_l T_l (\mathbf{u}_l - \mathbf{u}_h)$$

and the momentum exchange rate,

$$\mathbf{R}_{lh} = -\beta_0 n_l \nabla T_l - \alpha_0 \frac{n_l m_l}{\tau_{lh}}\left(\mathbf{u}_l - \mathbf{u}_h\right)$$

$$= m_l n_l \frac{d\mathbf{u}_l}{dt} + \nabla p_l + \nabla\cdot\boldsymbol{\pi}_l - Z_l e n_l \mathbf{E} \,. \quad (7)$$

Here the thermal conductivity is

$$\kappa_{lh} = \gamma_0 \frac{n_l T_l \tau_{lh}}{m_l} \,, \quad (8)$$

where

$$\tau_{lh} = \frac{3\sqrt{m_l}\,T_l^{3/2}}{4\sqrt{2\pi}\,e^4 n_h Z_l^2 Z_h^2 \ln \Lambda_{lh}} \,, \quad (9)$$

$\alpha_0 = 3\pi/32$, $\beta_0 = 3/2$, and $\gamma_0 = 128/3\pi$. The stress tensor π_l, which has been added to Eq. (7), will be subsequently evaluated. By analogy with electron-transport theory we identify α_0, β_0, and γ_0 as the resistivity, thermoelectric, and electron thermal diffusion coefficients, respectively (in the high-Z limit). (Note the extra $1/\sqrt{2}$ factor in our definition of τ_{lh}.) These results are in close agreement with the work of Hirshman,[3] who derived the thermal transport and momentum transfer coefficients numerically (via a Sonine polynomial expansion) for plasmas of arbitrary composition.

By comparing Eq. (1) with Eq. (8) we note a significant increase in the conductivity coefficient γ. Differences with regards to the mass and Z dependencies are also apparent. The ratio between the conductivities is given by

$$\frac{\kappa_{lh}}{\langle\kappa_i\rangle} = \frac{\gamma_0}{\gamma_i}\sqrt{\frac{\langle m_i\rangle}{2m_l}}\,\frac{\left\langle Z_i^2\right\rangle^2}{Z_l^2 Z_h^2}\,\frac{n_l}{n_h}\,\frac{\ln \Lambda_i}{\ln \Lambda_{lh}} \,. \quad (10)$$

If we consider a fully ionized CH plasma, where $m_l = m_p$ is the proton mass, $\langle m_i\rangle = 6.5 m_p$, $n_l = n_h$, $\left\langle Z_i^2\right\rangle = 18.5$, $Z_l^2 = 1$, and $Z_h^2 = 36$, we obtain $\kappa_{lh}/\langle\kappa_i\rangle \approx 60$. The thermal conductivity contribution from the h species is expected to be negligible since

$$q_h/q_l \sim \left(n_h/n_l\right)\left(m_l/m_h\right)^{1/2}\left(Z_l^2/Z_h^2\right) << 1.$$

Comparisons with the conductivity of a pure-H plasma (κ_l) and a pure-C plasma (κ_h) show that $\kappa_h : \kappa_{lh} : \kappa_l = 1 : 306 : 4500$.

The same type of analysis can be used to calculate the viscosity coefficient. The linearized Fokker-Planck equation describing the stress tensor contribution to the distribution function, which is now expanded as

$$f = f_0 + \mathbf{w}\cdot\mathbf{f}_1/w + \mathbf{ww} : \mathbf{f}_2/w^2 ,$$

is given by[2]

$$\mathbf{f}_2^l = \frac{w^4}{6n_h Y_{lh}} \frac{\partial f_0^l}{\partial w} \mathbf{U}_l , \qquad (11)$$

where

$$\mathbf{U}_l = \nabla \mathbf{u}_l + (\nabla \mathbf{u}_l)^T - \frac{2}{3}\mathbf{I}\nabla\cdot\mathbf{u}_l$$

is the rate-of-strain tensor of the l species (superscript T denotes the transpose and $\mathbf{I}$ is the unit dyadic). From the definition of the anisotropic part of the pressure tensor,

$$\boldsymbol{\pi}_l = -\eta_{lh}\mathbf{U}_l = \frac{8\pi m_l}{15}\int_0^\infty \mathbf{f}_2^l w^4 dw , \qquad (12)$$

we find that

$$\eta_{lh} = \mu_0 n_l T_l \tau_{lh} , \qquad (13)$$

where $\mu_0 = 256/45\pi \approx 1.81$. Note that this value of μ_0 extends the electron viscosity given by Braginskii [i.e., $\mu_0(Z=1) = 0.73$] to the high-Z limit.

As before, we can compare Eq. (13) with the averaged version of the standard formula [Eq. (2)] to obtain

$$\frac{\eta_{lh}}{\langle \eta_i \rangle} = \frac{\mu_0}{\mu_i}\sqrt{\frac{m_l}{2\langle m_i\rangle}}\frac{\langle Z_i^2\rangle^2}{Z_l^2 Z_h^2}\frac{n_l}{n_h}\frac{\ln\Lambda_i}{\ln\Lambda_{lh}} . \qquad (14)$$

Using the example of a CH plasma we then find that $\eta_{lh}/\langle \eta_i\rangle \approx 5$. The viscosity contribution from the h species is expected to be small since

$$\pi_h/\pi_l \sim (n_h/n_l)(m_h/m_l)^{1/2}\left(Z_l^2/Z_h^2\right) << 1 .$$

To illustrate the importance of these results we calculate the damping of collisional ion-acoustic waves in a CH plasma. Writing $\mathbf{u}_i = (\delta u_i, 0, 0)\exp(ikx - i\omega_i t)$ etc., the linearized single-species ion fluid equations, assuming collisionless and isothermal electrons, become

$$-\omega_i \delta n_i + n_i k \delta u_i = 0 , \qquad (15)$$

$$\omega_i m_i n_i \delta u_i = kT_i\delta n_i + k\delta T_i n_i + k\delta\pi_{ixx} + k\delta\phi n_i Z_i e , \qquad (16)$$

and

$$-\frac{3}{2}\omega_i n_i \delta T_i + kn_i T_i \delta u_i = -k\delta q_i . \qquad (17)$$

The perturbed electric potential, assuming quasi-neutrality and neglecting Landau damping, is given by $\delta\phi = \delta n_i T_e/en_i$.

Equations (15)–(17) yield a cubic dispersion relation, with roots ω_i corresponding to two counter-propagating and decaying ion-acoustic waves and a zero-frequency entropy wave (e.g., Ref. 4). By requiring strong collisionality, i.e., $\omega_i\tau_i << 1$, we are able to simplify the dispersion relation and obtain the following expressions for the ion-acoustic mode:

$$\frac{|\mathrm{Re}(\omega_i)|}{k} = v_i\sqrt{\frac{5}{3} + \frac{Z_i T_e}{T_i}} \equiv c_s , \qquad (18)$$

$$\frac{\mathrm{Im}(\omega_i)}{kv_i} = -\frac{2}{3}\left(\mu_i + \frac{\gamma_i}{5 + 3Z_i T_e/T_i}\right)k\lambda_i , \qquad (19)$$

where $v_i = (T_i/m_i)^{1/2}$ is the ion thermal velocity, $\lambda_i = v_i\tau_i$ is its mean free path, and c_s is the sound speed.

To generalize these results to a plasma with light and heavy ions we would strictly need separate fluid equations for each species, and the resulting dispersion relation would be a sixth-order polynomial in ω. Instead, however, we can use the fact that $|\mathrm{Im}(\omega)| << |\mathrm{Re}(\omega)|$ to calculate the damping directly from the rate of entropy production. This approach, described in detail by Braginskii,[1] is simpler than solving the dispersion relation and provides further physical insight into the damping processes.

The dissipative processes (thermal conduction, viscosity, and joule heating) are much weaker for the heavy particles than for the light, so we consider only the entropy of the light particles. We start with the entropy balance equation

$$\frac{\partial S_l}{\partial t} + \nabla \cdot \left(S_l \mathbf{u}_l + \frac{\mathbf{q}_l}{T_l} \right)$$

$$= \frac{1}{T_l}\left[-\frac{1}{2}\boldsymbol{\pi}_l : \mathbf{U}_l - \mathbf{q}_l \cdot \nabla \ln T_l + Q_{lh} \right], \tag{20}$$

where $S_l = s_l n_l$ is the specific entropy, and (to lowest order in τ_{lh}) $Q_{lh} = -\mathbf{R}_{lh} \cdot (\mathbf{u}_l - \mathbf{u}_h)$ represents the heating of the light particles resulting from collisions with the heavy particles. To calculate the damping of a small-amplitude sound wave, we define the average over the wavelength L by example

$$\overline{S_l} = \frac{1}{L}\int_0^L S_l dz \, . \tag{21}$$

Averaging Eq. (20) we then obtain an expression for the rate of entropy production in the wave:

$$\frac{d\overline{S}_l}{dt} = \frac{4}{3}\mu_0 n_l \tau_{lh} \overline{(\nabla \cdot \mathbf{u}_l)^2}$$

$$+\gamma_0 \frac{n_l \tau_{lh}}{m_l T_l} \overline{|\nabla T_l|^2} + \alpha_0 \frac{m_l n_l}{T_l \tau_{lh}} \overline{|\mathbf{u}_l - \mathbf{u}_h|^2} . \tag{22}$$

We represent the amplitude of the wave by

$$\delta n_l / n_l = \delta n_h / n_h = \xi \sin(kx - \omega t),$$

so that to lowest order in τ_{lh} we have

$$T_l = T_h \equiv T_i,$$

$$\delta T_l / T_l = 2/3 \xi \sin(kx - \omega t),$$

and

$$\delta u_l = \delta u_h = \xi c_s \cos(kx - \omega t) .$$

Using Eq. (7) to evaluate $\delta u_l - \delta u_h$ to first order in τ_{lh} then yields

$$\frac{d\overline{S}_l}{dt} = k^2 n_l \tau_{lh} \left[\frac{2}{3}\mu_0 c_s^2 + \frac{2}{9}\gamma_0 \frac{T_i}{m_l} + \frac{1}{2\alpha_0}\frac{T_i}{m_l} \left(\frac{2\beta_0}{3} + \frac{5}{3} + \frac{Z_l T_e}{T_i} - \frac{m_l c_s^2}{T_i} \right)^2 \right] \xi^2 . \tag{23}$$

The amplitude damping rate is given by

$$\mathrm{Im}(\omega) = -\frac{T_i}{2\overline{\varepsilon}}\frac{d\overline{S}_l}{dt}; \qquad \overline{\varepsilon} = \frac{1}{2} n_i m_i c_s^2 \xi^2 \ .$$

This expression gives the rate at which the energy of the wave, ε, is degraded to heat. The resulting damping rate formula is

$$\frac{\mathrm{Im}(\omega_{lh})}{k v_l} = -\frac{2}{3\left(5 + 3\langle Z_i \rangle T_e / T_i\right)} \left\{ \gamma_0 + \mu_0 \left(\frac{m_l}{\langle m_i \rangle} \right) \left(5 + 3\langle Z_i \rangle T_e / T_i \right) + \left(\frac{3}{2} \right)^2 \frac{1}{\alpha_0} \left[\frac{2\beta_0}{3} + \left(\frac{5}{3} + \frac{Z_l T_e}{T_i} \right) - \left(\frac{5}{3} + \frac{\langle Z_i \rangle T_e}{T_i} \right) \left(\frac{m_l}{\langle m_i \rangle} \right) \right]^2 \right\} \left(\frac{n_l}{n_i} \right) k \lambda_{lh} , \tag{24}$$

where $\lambda_{lh} = v_l \tau_{lh}$ and $v_l = (T_l / m_l)^{1/2}$. Here we can readily identify contributions due to thermal diffusion, viscosity, and joule heating by the coefficients γ_0, μ_0, and α_0, respectively.

An interesting feature of Eq. (24) is that it predicts the dominance of thermal diffusion over viscous effects for $\left(m_l / \langle m_i \rangle\right)\left(5 + 3\langle Z_i \rangle T_e / T_i\right) < 15/2$ (i.e., $T_e / T_i < 4.2$ for a

CH plasma, where $\langle Z_i \rangle = 3.5$ and $\langle m_i \rangle / m_l = 6.5$); whereas in the conventional formula the viscous damping is always dominant. More important, however, is the emergence of a joule-damping mechanism that is not present in the single-fluid model. It is easily shown for the case of CH plasma that this mechanism is dominant and at least three times larger than the thermal-diffusion mechanism. In terms of overall damping rate, a comparison between Eqs. (19) and (24), for CH, shows that

$$\frac{\mathrm{Im}(\omega_{lh})}{\mathrm{Im}(\langle \omega_i \rangle)} \approx \frac{101 + 34\, T_e/T_i + 2.77(T_e/T_i)^2}{0.86 + T_e/T_i}. \qquad (25)$$

This predicts an increase in the damping rate by at least a factor of 58.

Another interesting feature of Eq. (24) is that the joule-damping mechanism becomes independent of T_e/T_i for plasmas in which both species of ions have the same charge-to-mass ratio, so $\langle Z_i \rangle / \langle m_i \rangle = Z_l / m_l$. In such a plasma the electric field, which is the only mechanism by which the electron pressure can affect the ions, cannot drive a velocity difference in the two species, and so cannot contribute to joule heating. In this case, the viscous damping can eventually dominate for sufficiently large T_e/T_i.

In summary, the ion-transport coefficients have been calculated for a fully ionized unmagnetized plasma composed of light and heavy ions. The results show that using standard single-ion formulas with averaged ion masses and ionizations can lead to significant underestimations of the thermal conductivity, viscosity, and joule dissipation. The implications for the collisional damping of ion-acoustic waves are that joule heating and thermal diffusion can become the dominant damping mechanisms and the overall damping rate increases.

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