Effective Drift Velocity from Turbulent Transport by Vorticity

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Highlighted here are the differing roles of vorticity and strain in the transport of coarse-grained scalars at length-scales larger than l by smaller-scale (subscale or subgrid or unresolved) turbulence. We use the first term in a multiscale gradient expansion due to Eyink,[1] which exhibits excellent correlation with the exact subscale physics when the partitioning length l is any scale smaller than that of the spectral peak. We show that unlike subscale strain, which acts as an anisotropic diffusion/anti-diffusion tensor, subscale vorticity's contribution is solely a conservative advection of coarse-grained quantities by an eddy-induced non-divergent velocity, $\mathbf{v}_*$, that is proportional to the curl of vorticity. Therefore, material (Lagrangian) advection of coarse-grained quantities is accomplished not by the coarse-grained flow velocity, $\bar{\mathbf{u}}_l$, but by the effective velocity, $\bar{\mathbf{u}}_l + \mathbf{v}_*$, the physics of which may improve hydrodynamic modeling.

Basic considerations from fluid dynamics indicate that the distance between particles in a laminar flow is determined by the strain.[2] Vorticity merely imparts a rotation on their separation vector $\mathbf{r}$ without affecting its magnitude. This behavior can be seen by considering the velocity, $\mathbf{u}$, difference between particles P and Q at positions $\mathbf{x}$ and $\mathbf{x} + \mathbf{r}$, respectively,

$$\mathbf{u}_Q - \mathbf{u}_P = \delta\mathbf{u} = \mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x}) = \mathbf{r}\cdot\nabla\mathbf{u}\big|_{\mathrm{x}} + \ldots, \tag{1}$$

where a Taylor-series expansion is justified for short distances $|\mathbf{r}|$ over which the flow is sufficiently smooth. In the Lagrangian frame of P at $\mathbf{x}$, the separation from Q evolves as

$$\frac{D\mathbf{r}}{Dt} = \delta\mathbf{u} = \mathbf{r}\cdot\mathbf{S} + \underbrace{\mathbf{r}\cdot\boldsymbol{\Omega}}_{\frac{1}{2}\boldsymbol{\omega}\times\mathbf{r}}, \tag{2}$$

where the velocity gradient tensor, $\nabla\mathbf{u} = \mathbf{S} + \boldsymbol{\Omega}$, has been decomposed into the symmetric strain rate tensor $\mathbf{S} = \left[\nabla\mathbf{u} - (\nabla\mathbf{u})^T\right]/2$ and the antisymmetric vorticity tensor $\boldsymbol{\Omega} = \left[\nabla\mathbf{u} - (\nabla\mathbf{u})^T\right]/2 = -1/2\epsilon_{ijk}\omega_k$. Here, $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ is vorticity and ϵ_{ijk} is the Levi–Civita symbol. Taking an inner product of Eq. (2) with $\mathbf{r}$,

$$\frac{1}{2}\frac{D|\mathbf{r}|^2}{\mathrm{d}t} = \mathbf{r}\cdot\mathbf{S}\cdot\mathbf{r}, \tag{3}$$

shows that the distance is determined by the strain. Vorticity in Eq. (2) only acts to rotate $\mathbf{r}$ without changing its magnitude.

These considerations hinge on the critical assumption that the flow is sufficiently smooth over separation $\mathbf{r}$, which is patently invalid in a turbulent flow for $\mathbf{r}$ at inertial scales.[3] However, a version of this story survives due to the property of scale locality,[4] which justifies an expansion in scale. The main result of this research is Eq. (4),

$$\partial_t \overline{C}_l + \nabla \cdot \left[\left(\overline{\mathbf{u}}_l + \mathbf{v}_* \right) \overline{C}_l \right] = -\nabla \cdot \left[\mathbf{J}(C) \right], \tag{4a}$$

$$\mathbf{v}_* = \frac{1}{2} A l^2 \nabla \times \left(\nabla \times \overline{\mathbf{u}}_l \right), \tag{4b}$$

where Eq. (4a) shows us that coarse-grained simulations in general, including those from radiation-hydrodynamics inertial confinement fusion codes, may need to solve this equation to represent the unresolved (subgrid) vorticity physics self-consistently. Equation (4b) is an expression for the eddy-induced advection velocity $\mathbf{v}_*$ affecting length scales larger than l, which may be the grid cell size in a simulation. In Eq. (4a), $\mathbf{J}(C)$ can represent traditional subgrid models such as turbulent diffusion,[5] $\mathbf{J}(C) = -\alpha_{\text{turb}} \nabla \overline{C}_l$.

In summary, it is shown that unlike subscale strain, which acts as an anisotropic diffusion/anti-diffusion tensor, subscale vorticity's contribution at leading order is solely a conservative advection of coarse-grained scalars by an eddy-induced velocity $\mathbf{v}_*$ proportional to the curl of vorticity. Evidence of excellent agreement between the leading order terms and the exact ones from a 3-D compressible turbulence simulation are shown in Fig. 1. While the focus of this summary was on the transport of scalars, a similar analysis may also apply to the transport of momentum. Since the convergence of Eyink's expansion and, therefore, the dominance of the leading order term relies on ultraviolet scale locality,[4] these results and conclusions may not hold at length scales larger than those of the spectral peak. In other words, for coarse-grained simulations to use this modeling framework, they need to directly resolve the most energetic scales. Otherwise, some of the assumptions may not be valid. Note that the unresolved (subgrid) scales can still have the dominant vorticity contribution since energy and vorticity can occupy different scale ranges.

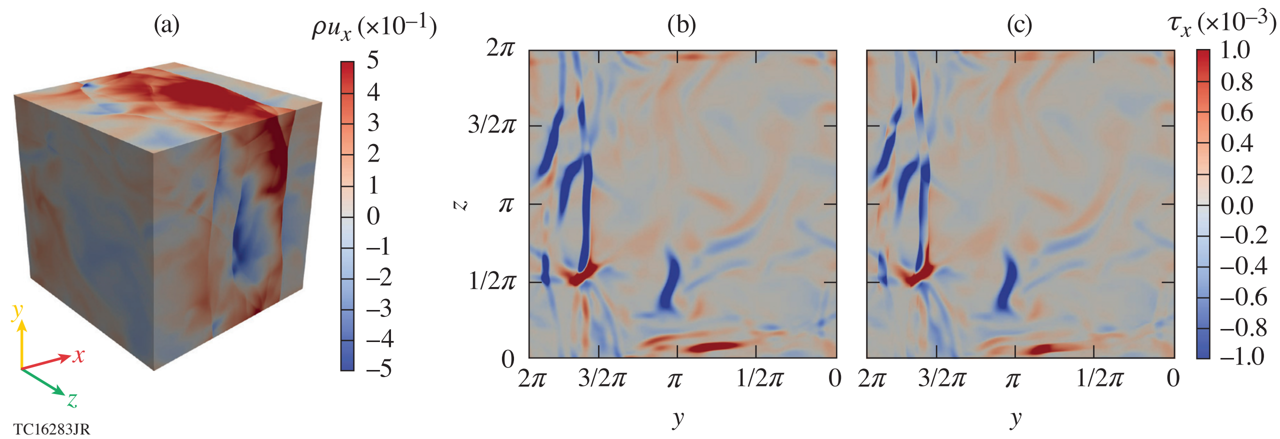

Figure 1
(a) Visualization of x component of momentum, ρu_x, in physical space from the 1024^3 compressible turbulence direct numerical simulation (DNS). The image is shown at an instant of time after the flow has reached steady state. Shocks can be seen as discontinuities. [(b),(c)] A 2-D slice at $x = 0$ from a snapshot of the 3-D compressible turbulence DNS, comparing the (b) exact $\overline{\tau}_l(u_x, \rho)$ at $l = 0.19635$ with (c) its approximation $\tau_m = 1/3 M_2 l^2 \partial_k \overline{\rho} \partial_k \overline{u}_x$.

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