Analytic Phase Solutions of Three-Wave Interactions

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Analytic solutions for three- and four-wave nonlinear interactions have long been available since the very early discussion of this phenomena.[1] The original approach provided amplitude and relative phase solutions as well as a general methodology based on utilizing constants of motion. The conservation laws involving the total and relative powers and relative phase between these waves provide an efficient way of solving the coupled-wave equations. The phase solution provided by Ref. 1 is the relative phase solution between interacting waves, but *no closed-form phase solution of an individual wave* has been presented over the last six decades. Smith and Bowers have shown that one of the interacting waves that starts from zero intensity always develops a phase that is linearly proportional to wave-vector mismatch.[2] Buchvarov *et al.* derived phase solutions by approximating the intensity solutions using an approximate form of the elliptic sinus function.[3] Marhic found analytic phase solutions for four-wave interactions but did not extend the work to three-wave mixing.[4] Ross *et al.* found phase solutions under nondepleted pump assumption.[5] The authors also provided the integral form of the phase solution as a function of pump depletion in a pump-depletion regime.[6] A survey of these works reveals that complete analytic phase solutions for three-wave mixing have not been previously published, while the necessity of the phase solutions has been mostly addressed by numerical methods.[7] In this summary, analytic phase solutions are shown to exist for three-wave mixing, expressible using standard mathematical functions. The solution provided by Ross *et al.*[6] does not explicitly show the functional dependence of phase on the wave-vector mismatch, Δk. It rather shows a linear dependence of Δk with residual terms hidden in the integral. The integral also does not provide an advantage in calculation speed over a direct numerical integration of differential equations.

Despite the availability of numerical approaches for directly solving differential equations with a minimum number of approximations, analytic solutions are still powerful tools in calculating optical parametric chirped-pulse–amplification (OPCPA) performance where the group delay and walk-off terms are often negligible. The temporal slice or the corresponding frequency slice of a stretched signal pulse interacts with the concurrent portion of the pump pulse, independent of other frequency slices. The local approximation approach allows one to apply analytic solutions at each wavelength slice. This computational efficiency allows for fast estimation and optimization of the phase performance in the amplified signal beam. Phase performance is becoming increasingly important in ultra-broadband, large-scale OPCPA systems since they are associated with the performance of pulse compression and focusing.

The equations for three interacting waves inside a second-order nonlinear medium in a normalized electric field (E') unit are written as

$$\frac{\mathrm{d}E'_\mathrm{s}}{\mathrm{d}z} = \mathrm{i}E'^*_\mathrm{i}E'_\mathrm{p}\exp\left(-i\Delta kz\right), \quad \frac{\mathrm{d}E'_\mathrm{i}}{\mathrm{d}z} = \mathrm{i}E'^*_\mathrm{s}E'_\mathrm{p}\exp\left(-\mathrm{i}\Delta kz\right), \quad \frac{\mathrm{d}E'_\mathrm{p}}{\mathrm{d}z} = \mathrm{i}E'_\mathrm{s}E'_\mathrm{i}\exp\left(i\Delta kz\right),$$

where the subscripts s, i, and p indicate signal, idler, and pump fields. The z axis is defined to be normal to the crystal surface and Δk is measured along the z axis, i.e., $\Delta k = k_{\mathrm{s},z} + k_{\mathrm{i},z} - k_{\mathrm{p},z}$. Normalized electric fields are defined as

$$E'_\mathrm{s} = \frac{d}{c}\sqrt{\frac{\omega_\mathrm{i}\omega_\mathrm{p}}{n_\mathrm{i}n_\mathrm{p}}}E_\mathrm{s} = \sqrt{u}\exp\left(i\phi_\mathrm{s}\right), \quad E'_\mathrm{i} = \frac{d}{c}\sqrt{\frac{\omega_\mathrm{s}\omega_\mathrm{p}}{n_\mathrm{s}n_\mathrm{p}}}E_\mathrm{i} = \sqrt{v}\exp\left(i\phi_\mathrm{i}\right), \quad E'_\mathrm{p} = \frac{d}{c}\sqrt{\frac{\omega_\mathrm{s}\omega_\mathrm{i}}{n_\mathrm{s}n_\mathrm{i}}}E_\mathrm{p} = \sqrt{w}\exp\left(i\phi_\mathrm{p}\right).$$

The constants d and c are the second-order effective nonlinear coefficient and speed of light. It can be shown that there are four constants of motion: $p = u + w$, $q = v + w$, $r = u - v$, and $s = \sqrt{uvw}\cos\left(\Delta kz + \phi_\mathrm{s} + \phi_\mathrm{i} + \phi_\mathrm{p}\right) - (1/2)\Delta k w_0$. Since these are constants, we can set these values using the initial intensity and phase values, e.g.,

$$p = u_0 + w_0,\ s = \sqrt{u_0 v_0 w_0}\cos\left(\phi_{\mathrm{s},0} + \phi_{\mathrm{i},0} - \phi_{\mathrm{p},0}\right) - \frac{1}{2}\Delta k w_0, \text{ etc.}$$

The 0 subscript indicates the initial value at $z = 0$.

The solutions in the case of $v_0 = 0$ (no idler input) will be shown here without derivation. This case covers the most-practical case for optical parametric amplification. The intensity solutions are

$$w(z) = w_c - \frac{w_c - w_0}{\mathrm{dn}^2\left(z/z_d, m\right)}, \quad u(z) = p - w(z), \quad v(z) = q - w(z),$$

where $z_d = 1/\sqrt{w_c - w_a}$ and $m = (w_b - w_a)/(w_c - w_a)$. The three pump parameters (w_a, w_b, w_c) are calculated as

$$w_a = (1/2)\left(p' - \sqrt{p'^2 - \Delta k^2 w_0}\right), \quad w_b = w_0, \quad \text{and } w_c = (1/2)\left(p' + \sqrt{p'^2 - \Delta k^2 w_0}\right),$$

with $p' = p + \Delta k^2/4$. The function dn is one of the Jacobi elliptic functions. The unit of normalized intensity is in (distance unit of z)$^{-2}$. The pump depletion distance, which is of the order of z_d, is easily estimated from the initial normalized pump intensity or vice versa.

The phase solutions are

$$\phi_\mathrm{s}(z) = \phi_\mathrm{s}(0) - \frac{\Delta k}{2}\left(\frac{w_c - w_0}{w_c - p}\right)\left\{z - z_d \Pi\left[\frac{(w_c - p)}{-u_0}m; am(z/z_d, m), m\right]\right\},$$

$$\phi_\mathrm{i}(z) = \pi/2 - \phi_\mathrm{s}(0) + \phi_\mathrm{p}(0) - \frac{1}{2}\Delta kz,$$

$$\phi_\mathrm{p}(z) = \phi_\mathrm{p}(0) + \frac{\Delta k}{2}\left(\frac{w_c - w_0}{w_c}\right)\left\{z - z_d \Pi\left[\frac{w_\mathrm{c}}{w_0}m; am(z/z_d, m), m\right]\right\}. \tag{1}$$

Here, $\Pi(.;.,.)$ and $am(.,.)$ are an elliptic integral of the third kind and the Jacobi amplitude function, respectively.

The signal gain $[u(z)/u(0)]$ and phase plots are shown in Fig. 1 for a 6.8-cm KDP crystal with its c axis oriented at 41.06° with respect to the z axis. The initial signal and pump intensities are 1 MW/cm^2 at 0.930 μm and 1 GW/cm^2 at 0.527 μm, respectively. The phase plot shows linear behavior within the amplification bandwidth, which can be approximated to $-\Delta k/2\left[z - \tanh\left(\sqrt{w_0}z\right)/\sqrt{w_0}\right]$ (dashed line). In the depletion regime for a 20-dB amplifier, $\sqrt{w_0}z \sim 3.68$, so phase in the linear regime can be further approximated to $-0.36\,\Delta kz$. Outside the gain bandwidth the phase asymptotically approaches $-w_0 z/\Delta k$ (dotted line). The signal beam in this regime therefore has a pump-beam-intensity–dependent phase profile, which can be used to produce an instantaneous phase profile. This behavior is similar to the cascaded nonlinearity.[8]

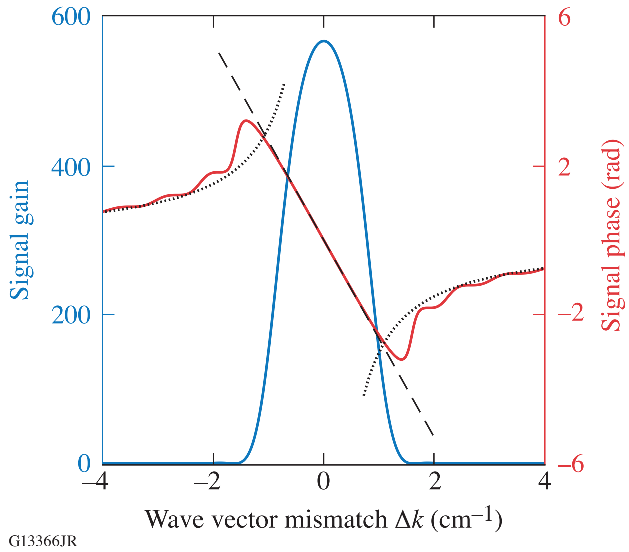

Figure 1
Intensity and phase response with respect to wave-vector mismatch Δk.

These results provide convenient tools to estimate the phase performance of an OPA in either spatial or temporal domains. The analytic phase solutions for the case of nonzero idler input and for the case of sum-frequency and second-harmonic generation are discussed in detail in Ref. 9.

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