Scaling of Turbulent Viscosity and Resistivity: Extracting a Scale-Dependent Turbulent Magnetic Prandtl Number

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It is widely believed that turbulence plays an important role in the outward transport of angular momentum in accretion disks for inward mass accretion.[1] The simplest conceptual framework is to think of turbulence as an effective (or turbulent) viscosity v_t, which leads to the "turbulent diffusion" of angular momentum at scales far larger than viscous scales; this description has long shaped our thinking of accretion disk dynamics.[2] Similarly, magnetic fields can be transported outward by an effective (or turbulent) resistivity η_t. In this way, the magnetic-field configuration in accretion disks may be influenced by a balance between the inward advection by accretion and the outward diffusion by turbulent resistivity. This balance between the competing effects of v_t and η_t is captured by the turbulent magnetic Prandtl number $\mathrm{Pr}_t \equiv v_t/\eta_t$.

In this summary, we develop a new approach to determine these "effective transport" coefficients acting at different length scales. To our knowledge, our work is the first to calculate the turbulent magnetic Prandtl number as a function of length scale. Our results indicate that it has values of ≈1 to 2 at the smallest ideal hydrodynamic scales, increasing to ≈5 to 10 at the largest scales.

We analyze the coarse-grained kinetic energy (KE) and magnetic energy (ME) density balance of the incompressible magnetohyrodynamic (MHD) equations[3] and extract the effective transport coefficients v_t, η_t, and Pr_t as a function of length scale. We analyze the energy budgets resulting from the eddy viscosity model. Within our coarse-graining framework, this is equivalent to having the rate of energy cascading to scales smaller than l equal to a turbulent dissipation acting on scales $>\ell$:

$$2v_t\left\langle\left|S_\ell\right|^2\right\rangle \equiv \langle\overline{\Pi}_\ell^u\rangle, \eta_t\left\langle\left|\overline{\mathbf{J}}_\ell\right|^2\right\rangle \equiv \langle\overline{\Pi}_\ell^b\rangle. \tag{1}$$

It is possible to relate v_t and η_t to energy spectra. Indeed, the space-averaged turbulent dissipation can be expressed in terms of energy spectra:

$$\langle\overline{\Pi}_\ell^u\rangle = 2v_t\left\langle\left|\overline{\mathbf{S}}_\ell\right|^2\right\rangle = 2v_t\int_0^k k'^2 E^u(k')\mathrm{d}k, \langle\overline{\Pi}_\ell^b\rangle = \eta_t\left\langle\left|\overline{\mathbf{J}}_\ell\right|^2\right\rangle = 2\eta_t\int_0^k k'^2 E^b(k')\mathrm{d}k'. \tag{2}$$

The kinetic and magnetic energy spectra scale as

$$E^u(k) \propto k^{-2\sigma_u-1},\ E^b(k) \propto k^{-2\sigma_b-1}. \tag{3}$$

The exponents σ_u and σ_b in the scaling of spectra are related to the scaling of velocity and magnetic-field increments:[3]

$$\delta u(\ell) \propto \ell^{\sigma_u},\ \delta B(\ell) \propto \ell^{\sigma_b}. \tag{4}$$

For sufficiently high Reynolds number flows, X. Bian and H. Aluie[4] showed that $\langle \overline{\Pi}_\ell^u \rangle$ and $\langle \overline{\Pi}_\ell^b \rangle$ become constant, independent of scale in the so-called "decoupled range." From definitions in Eq. (1), and considering the scaling relations discussed above, we can infer that the turbulent transport coefficients vary with scale in the decoupled range as follows:

$$v_t \propto k^{-2(1-\sigma_u)},\ \eta_t \propto k^{-2(1-\sigma_b)},\ \mathrm{Pr_t} \propto k^{-2(\sigma_b-\sigma_u)}. \tag{5}$$

We conducted pseudo-spectral direct numerical simulations of MHD turbulence using hyperdiffusion with grid resolutions up to 2048^3. Figure 1 shows the effective transport coefficients as a function of scale calculated using their respective definitions in Eq. (1). We can see that $v_t(k) \sim k^{-5/3}$ and $\eta_t(k) \sim k^{-4/3}$ are consistent with relation in Eq. (5) when $\sigma_u = 1/6$ and $\sigma_b = 1/3$, as in our simulations. Moreover, we see that $\mathrm{PR_t}(k) \sim k^{1/3}$, which is also consistent with the derived scaling in Eq. (5) with $\sigma_u = 1/6$ and $\sigma_b = 1/3$ in our simulated flows. For accretion disks, this indicates that the flow may be more efficient at accreting large-scale magnetic fields radially inward than diffusing them outward.

Analyzing the kinetic and magnetic energy cascade rates, we infer power-law scaling in Eq. (5) for v_t, η_t, and $\mathrm{PR_t}$ given our definitions of those transport coefficients. This approach circumvents relying on particular values for the spectral scaling exponents (σ_u and σ_b) from a specific MHD phenomenology—whether it exists or not—by relying on results from X. Bian and H. Aluie[4] of conservative KE and ME cascades.

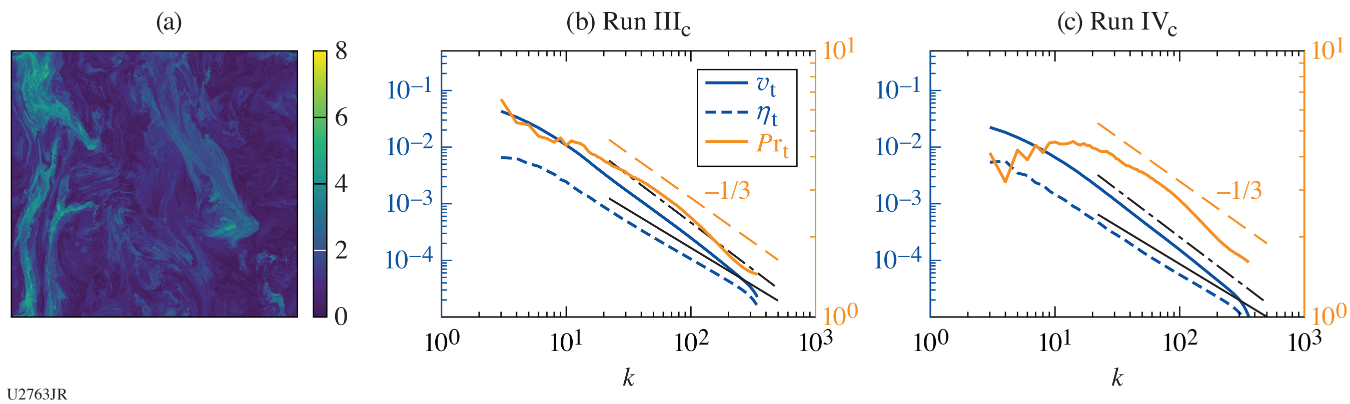

Figure 1
(a) A 2-D slice from a 3-D simulation of turbulent flow subject to a strong external magnetic field in the vertical. It highlights the anisotropic and complex transport in such flows. [(b),(c)] Plots of v_t, η_t, and $\mathrm{Pr_t}$ at different scales for two cases at the highest resolution. Three reference lines with a slope of –1/3, –5/3 (dashed–dotted black curve), and –4/3 (solid black curve) are added.

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