Temporal Reflection and Refraction of Optical Pulses Inside a Dispersive Medium: An Analytic Approach

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Reflection of electromagnetic waves at a temporal boundary has attracted considerable attention in several contexts over the last 20 years.[1] Most of these studies have focused on a nondispersive medium and ignored the frequency dependence of the refractive index on each side of the temporal boundary. The dispersive effects were included in a 2015 study[2] that considered the reflection and refraction of optical pulses at a moving boundary.

We develop an analytic approach for reflection of light at a temporal boundary inside a dispersive medium and derive frequency-dependent expressions for the reflection and transmission coefficients. Using the analytic results, we study the temporal reflection of an optical pulse and show that our results agree fully with a numerical approach used earlier. Our approach provides approximate analytic expressions for the electric fields of the reflected and transmitted pulses; where the width of transmitted pulse is modified, the reflected pulse is a mirrored version of the incident pulse. When a part of the incident spectrum lies in the region of total internal reflection, both the reflected and transmitted pulses are considerably distorted.

We consider propagation of optical pulses inside a dispersive medium (such as an optical fiber) with the propagation constant $\beta(\omega)$. We assume that the pulse's spectrum is relatively narrow (quasi-monochromatic approximation), and we can expand $\beta(\omega)$ around a reference frequency in a Taylor series as

$$\beta(\omega) = \beta_0 + \beta_1(\omega - \omega_0) + \frac{1}{2}\beta_2(\omega - \omega_0)^2, \tag{1}$$

where ω_0 is a reference frequency close to the central frequency of the pulse and we neglected the third- and higher-order dispersive terms. We assume that the pulse is approaching a temporal boundary moving at the speed v_B. The refractive index changes across this boundary by a constant amount Δn and the propagation constant after the boundary becomes

$$\beta_t(\omega) = \beta_0 + \beta_B + \beta_1(\omega - \omega_0) + \frac{1}{2}\beta_2(\omega - \omega_0)^2, \tag{2}$$

where $\beta_B = (\omega_0/c)\Delta n$.

The evolution of the pulse across the temporal boundary is governed by the following Eq. (2) satisfied by the slowly varying envelope $A(z,t)$ of the pulse:

$$\frac{\partial A}{\partial z} + \Delta\beta_1 \frac{\partial A}{\partial t} + i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2} = i\beta_B H(t - T_B)A, \tag{3}$$

where we work in a frame in which the temporal boundary appears stationary, i.e., $t = t' - z/v_B$, where t' is the time in the laboratory frame; $\Delta\beta_1 = \beta_1 - 1/v_B$ is the relative group delay of the pulse in this time frame. In Eq. (3), $H(t - T_B)$, is the heaviside step function taking the value 1, when $t > T_B$, and 0, when $t < T_B$. Given the initial pulse shape $A(0,t)$, numerical solutions of Eq. (3) show that the pulse splits into two parts moving at different speeds because of the spectral shifts induced at the boundary.[2]

Our objective is to solve Eq. (3) analytically. The numerical solutions of Eq. (3) indicate that any input pulse is partially reflected and partially transmitted at the temporal boundary such that the two parts have spectra shifted from that of the input pulses. We expect the same to occur for a plane wave. In other words, a plane wave at the frequency $\omega_0 + \Delta\omega$ is also reflected and transmitted at the boundary with different frequency shifts such that the slowly varying amplitude takes the form:

$$A = \begin{cases} e^{i(\beta' z - \Delta\omega t)} + R e^{i(\beta'_r z - \Delta\omega_r t)}, & t < T_B \\ T e^{i(\beta'_t z - \Delta\omega_t t)}, & t > T_B \end{cases}, \tag{4}$$

where R and T are the reflection and transmission coefficients that depend on $\Delta\omega$. Here $\Delta\omega$ is the frequency shift of the input plane wave from the reference frequency ω_0, and $\Delta\omega_r$ and $\Delta\omega_t$ are frequency shifts of the reflected and transmitted plane waves, respectively. These frequency shifts depend on $\Delta\omega$.

We note that Eq. (4) does not violate causality because it is based on plane waves and the time variable $t = t' - z/v_B$ is defined in a moving frame with t' representing the real time. Causality requires only that the wave packets, representing the reflecting and transmitted parts of the incident pulse, form only after the pulse has arrived at the temporal boundary located at $z_B = T_B / \Delta\beta_1$. As discussed later, this is indeed the case.

We find the frequency shifts $\Delta\omega_r$ and $\Delta\omega_t$ by substituting the solution in Eq. (4) into Eq. (3) for $z < z_B$ and $z > z_B$. This yields the following relations:

$$\begin{cases} \beta'(\Delta\omega) = \Delta\beta_1 \Delta\omega + \dfrac{\beta_2}{2}\Delta\omega^2 \\ \beta'_r(\Delta\omega_r) = \Delta\beta_1 \Delta\omega_r + \dfrac{\beta_2}{2}\Delta\omega_r^2 \\ \beta'_t(\Delta\omega_t) = \beta_B + \Delta\beta_1 \Delta\omega_t + \dfrac{\beta_2}{2}\Delta\omega_t^2 \end{cases}. \tag{5}$$

These are the dispersion relation in the moving frame. From Eq. (3), $A(z,t)$ should be continuous for all values of z. This happens when the three propagation constants are equal, i.e.,

$$\beta' = \beta'_r = \beta'_t. \tag{6}$$

As discussed in Ref. 2, these conditions result from the conservation of momentum in the moving frame. Combining Eqs. (5) and (6), we find two quadratic equations whose solutions determine $\Delta\omega_r$ and $\Delta\omega_t$ for a given $\Delta\omega$. The solution for $\Delta\omega_r$ is

$$\Delta\omega_r = -\frac{2\Delta\beta_1}{\beta_2} - \Delta\omega. \tag{7}$$

The solution for $\Delta\omega_t$ is a more complicated and is given by[2]

$$\Delta\omega_t = -\frac{\Delta\beta_1}{\beta_2} + \frac{1}{\beta_2}\sqrt{(\Delta\beta_1 + \beta_2\Delta\omega)^2 - 2\beta_2\beta_B}. \tag{8}$$

To find the reflection and transmission coefficients, R and T, we make use of the temporal boundary conditions at $t = T_B$. Specifically, we demand that both A and $\partial A/\partial t$ are continuous across the time boundary for any z. This requirement results in the following two equations:

$$\begin{aligned} &e^{i(\beta' z - \Delta\omega T_B)} + Re^{i(\beta'_r z - \Delta\omega_r T_B)} = Te^{i(\beta'_t z - \Delta\omega_t T_B)}, \\ &-i\Delta\omega e^{i(\beta' z - \Delta\omega T_B)} - i\Delta\omega_r Re^{i(\beta'_r z - \Delta\omega_r T_B)} = -i\Delta\omega_t Te^{i(\beta'_t z - \Delta\omega_t T_B)}. \end{aligned} \tag{9}$$

Using $\beta' = \beta'_r = \beta'_t$ from Eq. (6), we obtain the following analytic expressions for R and T:

$$\begin{cases} R(\Delta\omega) = \dfrac{\Delta\omega_t - \Delta\omega}{\Delta\omega_r - \omega_t} e^{i(\Delta\omega_r - \Delta\omega)T_B} \\ T(\Delta\omega) = \dfrac{\Delta\omega_r - \Delta\omega}{\Delta\omega_r - \omega_t} e^{i(\Delta\omega_t - \Delta\omega)T_B} \end{cases}. \tag{10}$$

These expressions contain a linear phase shift that depends on the boundary's location T_B. This phase shift is not important and can be removed by choosing $T_B = 0$; however, R and T can still be complex quantities. Figure 1 shows how their moduli and phases vary as a function of the physically measured frequency shift $\Delta\nu = (\Delta\omega/2\pi)$ using the notation $R = |R|e^{i\phi(R)}$ and $T = |T|e^{i\phi(T)}$. The parameters used in Fig. 1 are appropriate for an optical fiber acting as a dispersive medium[2] and have values $\Delta\beta_1 = 0.1$ ps/m, $\beta_2 = 5$ ps^2/km, and $\beta_B = 0.5$ m^{-1}.

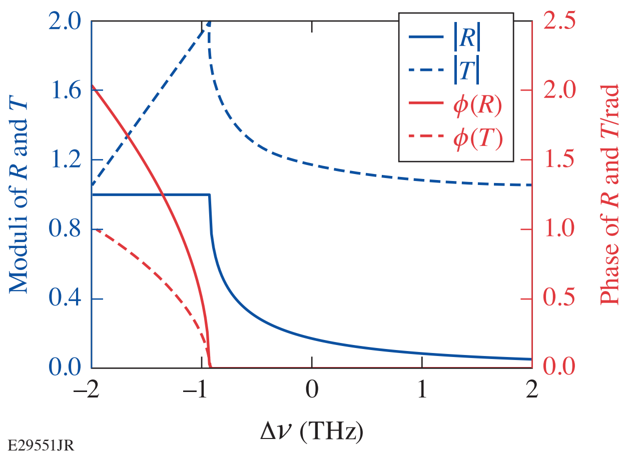

Figure 1
Frequency dependence of the reflection and transmission coefficients for $\Delta\beta_1 = 0.1$ ps/m, $\beta_2 = 5$ ps^2/km, and $\beta_B = 0.5$ m^{-1}. The solid blue curve represents the modulus of reflected wave and the solid red curve is the phase of the reflected wave. Likewise, the dashed blue curve is the modulus of the transmitted wave and the dahed red curve is the phase of the transmitted wave.

The most-striking feature in Fig. 1 occurs near $\Delta\nu_c = \sqrt{2\beta_2\beta_B} - \Delta\beta_1/2\pi\beta_2 = -0.93$ THz. When $\Delta\nu > \Delta\nu_c$, both R and T are real quantities; when $\Delta\nu < \Delta\nu_c$, they become complex. The reason for this change is related to the form of Eqs. (7) and (8). While $\Delta\omega_r$ is always real, $\Delta\omega_t$ can be complex depending on the sign of the discriminant in Eq. (8). The condition for this to occur is given by $(\Delta\beta_1 + \beta_2\Delta\omega)^2 < 2\beta_2\beta_B$. In this situation, $\Delta\omega_t$ becomes complex and the transmitted wave becomes evanescent. It can be shown that $|R| = 1$ holds for $\Delta\omega < \Delta\omega_c$. This is the temporal analog of total internal reflection discussed in Ref. 2. We call $\Delta\nu_c$ the critical frequency.

The preceding discussion applies to each specific frequency component of a pulse. We can use it to study how an incident pulse gets reflected and transmitted at the temporal boundary. Consider an incident pulse with the slowly varying amplitude

$$A(z = 0, t) = A_{in}(t). \tag{11}$$

We can decompose it into plane waves of different frequencies using the Fourier transform:

$$\tilde{A}(\Delta\omega) = \int A_{\text{in}}(t) e^{i\Delta\omega t} \mathrm{d}t. \tag{12}$$

The evolution of each plane-wave component is governed by Eq. (4). The total field can be calculated by integrating over the input pulse's spectrum to obtain:

If $t < T_{\text{B}}$,

$$\begin{aligned} A(z,t) &= \frac{1}{2\pi} \int \tilde{A}(\Delta\omega) e^{i\left[\beta'(\Delta\omega)z - \Delta\omega t\right]} \mathrm{d}\Delta\omega \\ &+ \frac{1}{2\pi} \int \tilde{A}(\Delta\omega) R(\Delta\omega) e^{i\left[\beta'_{\text{r}}(\Delta\omega)z - \Delta\omega_{\text{r}} t\right]} \mathrm{d}\Delta\omega. \end{aligned} \tag{13}$$

If $t > T_{\text{B}}$,

$$A(z,t) = \frac{1}{2\pi} \int \tilde{A}(\Delta\omega) T(\Delta\omega) e^{i\left[\beta'_{\text{t}}(\Delta\omega_{\text{t}})z - \Delta\omega_{\text{t}} t\right]} \mathrm{d}\Delta\omega. \tag{14}$$

This being our main result, it can be used to find the shapes and spectra of the reflected and transmitted parts of any input pulse.

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