Transport Coefficients for Magnetic-Field Evolution in Inviscid Magnetohydrodynamics

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In a magnetized plasma, the resistivity and electrothermal coefficient become tensors. Transport differs parallel and perpendicular to the magnetic field; when transport perpendicular to both the conventional direction of transport and the magnetic field occurs, it is called "cross-field transport." Braginskii[1] expressed the resulting Ohm's law in the form

$$\vec{E} = -\vec{v} \times \vec{B} + \frac{\vec{j}}{n_\mathrm{e} e} \times \vec{B} - \frac{\nabla P_\mathrm{e}}{n_\mathrm{e} e}$$
$$+ \frac{\eta_\parallel}{\varepsilon_0 \omega_\mathrm{pe}^2 \tau_\mathrm{ei}} \vec{b}\left(\vec{b} \cdot \vec{j}\right) + \frac{\eta_\perp}{\varepsilon_0 \omega_\mathrm{pe}^2 \tau_\mathrm{ei}} \vec{b} \times \left(\vec{j} \times \vec{b}\right) - \frac{\eta_\wedge}{\varepsilon_0 \omega_\mathrm{pe}^2 \tau_\mathrm{ei}} \left(\vec{b} \times \vec{j}\right)$$
$$- \beta_\parallel \left(\vec{b} \cdot \nabla T_\mathrm{e}\right) - \beta_\perp \vec{b} \times \left(\nabla T_\mathrm{e} \times \vec{b}\right) - \beta_\wedge \left(\vec{b} \times \nabla T_\mathrm{e}\right), \tag{1}$$

where v is ion velocity (m s^{-1}), B is magnetic field (T), j is current density (A m^{-2}), n_e is electron density (m^{-3}), e is electron charge (C), P_e is electron pressure (Pa), ε_0 is the permittivity of free space (F m^{-1}), ω_pe is electron plasma frequency (s^{-1}), τ_ei is electron–ion collision time (s), b is a unit vector in the direction of the magnetic field, T_e is electron temperature (eV), η is a dimensionless resistivity coefficient, β is a dimensionless electrothermal coefficient, $\parallel$ indicates parallel, $\perp$ indicates perpendicular, and $\wedge$ indicates cross. Electron viscosity has been neglected for simplicity. The only contributions the magnetized transport coefficients make to magnetohydrodynamics (MHD) are to modify magnetic-field evolution, given by $\nabla \times E$, and Ohmic heating, given by $j \cdot E$. The resulting expression for magnetic-field evolution is long and obscures the physical significance of the transport terms. Using vector identities, the equation can be rearranged into several different forms that are more compact and have terms with a clear physical interpretation.[2] The most compact form is

$$\frac{\partial \vec{B}}{\partial t} = \nabla \beta_\parallel \times \nabla T_\mathrm{e} + \frac{\nabla P_\mathrm{e} \times \nabla n_\mathrm{e}}{n_\mathrm{e}^2 e} + \nabla \times \left(\vec{v}_\mathrm{eff} \times \vec{B}\right) + \vec{v}_\eta \times \left(\nabla \times \vec{B}\right) + \eta_\parallel \frac{\delta_\mathrm{e}^2}{\tau_\mathrm{ei}} \nabla^2 \vec{B},$$
$$\vec{v}_\mathrm{eff} = \vec{v} - \frac{\vec{j}}{n_\mathrm{e} e}\left(1 + \frac{\eta_\wedge}{\chi_\mathrm{e}}\right) - \frac{\eta_\perp - \eta_\parallel}{\chi_\mathrm{e}} \frac{\vec{b} \times \vec{j}}{n_\mathrm{e} e} - \frac{\beta_\wedge}{\chi_\mathrm{e}} \frac{e\tau_\mathrm{ei}}{m_\mathrm{e}} \nabla T_\mathrm{e} + \frac{\beta_\parallel - \beta_\perp}{\chi_\mathrm{e}} \frac{e\tau_\mathrm{ei}}{m_\mathrm{e}} \nabla T_\mathrm{e} \times \vec{b}, \tag{2}$$
$$\vec{v}_\eta = -\nabla \left(\eta_\parallel \frac{\delta_\mathrm{e}^2}{\tau_\mathrm{ei}}\right), \quad \delta_\mathrm{e} = \frac{c}{\omega_\mathrm{pe}},$$

where $\chi_\mathrm{e} = eB\tau_\mathrm{ei}/m_\mathrm{e}$ is the electron Hall parameter, a dimensionless measure of the strength of magnetization compared to collisions. In this form, the transport coefficients are seen to lead to a magnetic-field source term $\nabla \beta_\parallel \times \nabla T$, advection of magnetic field with electron transport, seen in the terms dependent on j and ∇T_e in the effective velocity v_{eff}, advection of magnetic field to regions of lower resistivity from v_η, and magnetic diffusion. These physical effects are present in whatever mathematical form is chosen. Haines[3] attributes the advection terms to electron heat flux, which has terms in j as well as ∇T_e, describing the magnetic field as being frozen to the electrons responsible for thermal transport, not the bulk fluid, due to their lower collision frequency. In this analogy, the $\eta_\wedge / \chi_e$ term arises from perpendicular electrothermal heat flux, the $(\eta_\perp - \eta_\parallel)/\chi_e$ term from cross-field electron heat flux, the $\beta_\wedge / \chi_e$ term from perpendicular thermal conduction, and the $(\beta_\parallel - \beta_\perp)/\chi_e$ term from cross-field thermal conduction.

The advection terms in the effective velocity depend on modified transport coefficients that have not been explicitly considered in determining fits for the transport coefficients, motivating a reconsideration of these fits. We examined the fits given by Braginskii,[1] Epperlein and Haines,[4] and Ji and Held.[5] The transport coefficients are obtained by solving the Fokker–Planck equation in the limit of small mean free path and collision time. Braginskii used a third-order expansion in Laguerre polynomials and gives fits in χ_e for effective atomic numbers $Z = 1$, 2, 3, 4, and $Z \to \infty$ (no electron–electron collisions) stated to be within 20% of the approximate solutions. Braginskii's fits have incorrect limiting forms for $\eta_\wedge$ and $\beta_\perp$ as $\chi_e \to \infty$ and do not adequately constrain the values for $Z > 4$, but are still widely used. Epperlein and Haines used a direct numerical solution and give fits in χ_e for $Z = 1$, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14, 20, 30, 60, and $Z \to \infty$ accurate to within 15%. While Epperlein and Haines' results do allow a more-accurate interpolation of the coefficients for arbitrary Z than Braginskii's, their fitting parameters for $\eta_\perp, \eta_\wedge$, and $\beta_\perp$ are discontinuous in Z, so they can give physically incorrect gradients in these coefficients due to variations in Z, which will affect magnetic-field advection. Epperlein and Haines' fits are perhaps the most widely used in MHD codes. Ji and Held used the expansion in Laguerre polynomials, increasing the number of terms until the coefficients changed by less than 1%, which required up to 160 terms. They give fits in χ_e and Z, valid for any Z from 0 to 100, accurate to better than 1%, making them the most convenient and accurate set of fits of which we are aware. It is important to note that the stated fitting accuracies do not apply to the modified coefficients appearing in the effective velocity.

For the modified transport coefficients, Braginskii's incorrect limiting forms for $\eta_\wedge$ and $\beta_\perp$ as $\chi_e \to \infty$ are irrelevant. All three fits give $1 + \eta_\wedge / \chi_e \to 1$ and $(\beta_\parallel - \beta_\perp)/\chi_e \to \beta_\parallel / \chi_e$ as $\chi_e \to \infty$. There is good agreement on $\beta_\parallel$ and $\eta_\parallel$. The fits for $1 + \eta_\wedge / \chi_e$ agree to better than 5%. There is also good agreement on $\beta_\wedge / \chi_e$, which determines the Nernst velocity, except at small Hall parameters (<2) where Braginskii underestimates this term by up to 10%, which can be physically significant. Serious issues arise with the $(\eta_\perp - \eta_\parallel)/\chi_e$ and $(\beta_\parallel - \beta_\perp)/\chi_e$ terms, where both Epperlein and Haines and Ji and Held give physically incorrect results for $\chi_e \to 0$, as can be seen in Fig. 1. These modified transport coefficients represent cross-field transport of the magnetic field so they should be proportional to χ_e as $\chi_e \to 0$. Epperlein and Haines give quite the opposite behavior with a peak in both coefficients at $\chi_e < 1$, which increases with Z. Ji and Held's fits fail to go to zero as $\chi_e \to 0$, the error being most significant for $(\beta_\parallel - \beta_\perp)/\chi_e$, which goes negative and becomes increasingly negative as Z increases. Only Braginskii gives physically accurate results. Since Epperlein and Haines and Ji and Held are accurate for sufficiently large Hall parameters, we can see that Braginskii's $(\eta_\perp - \eta_\parallel)/\chi_e$ is a significant overestimate for moderate Hall parameters; it also has a nonphysical double hump.

To evaluate the accuracy of the fits for these modified coefficients, we obtained $(\beta_\perp - \beta_\parallel)/\chi_e$ from a direct numerical solution of the Fokker–Planck equation using the code *OSHUN*,[6] which is shown as squares in Fig. 1(a). Braginskii is the most accurate for Hall parameters close to zero, but Ji and Held rapidly give the most-accurate fit as the Hall parameter increases. We also compared the fits for the Nernst term $\beta_\wedge / \chi_e$ to the *OSHUN* results and found that Ji and Held's fit is the most accurate. For the $(\eta_\perp - \eta_\parallel)/\chi_e$ term, we used the results directly from Ji and Held's 320-term solution, shown as squares in Fig. 1(b), which again shows that Braginskii is the most accurate for Hall parameters close to zero, but Ji and Held rapidly give the most-accurate fit as the Hall parameter increases. Since none of the fits are accurate for $(\eta_\perp - \eta_\parallel)/\chi_e$ and $(\beta_\parallel - \beta_\perp)/\chi_e$, we tried fitting the direct calculations of these coefficients shown in Fig. 1 with a function of χ_e and Z based on those used by Ji and Held. The fits are too long to be included in this summary but can be found elsewhere.[2]

In conclusion, we have found that there are multiple magnetic-field advection terms that arise from the resistivity and electrothermal tensors in a magnetized plasma. These advection terms depend on significantly modified transport coefficients, which

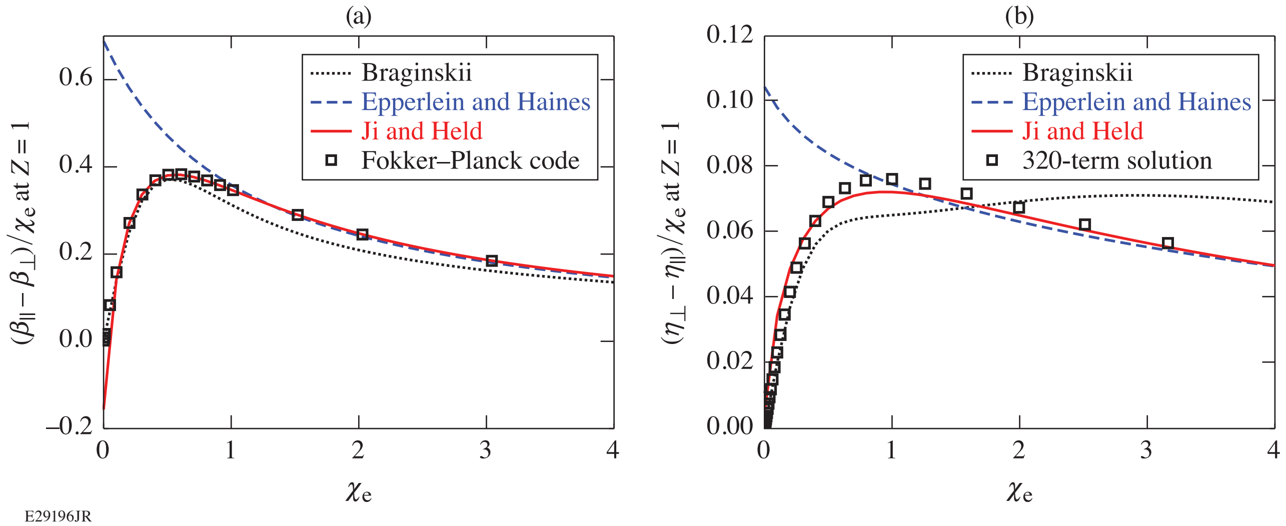

Figure 1
Modified (a) electrothermal and (b) resistivity coefficients that determine cross-field advection of magnetic field at $Z = 1$. Fits from Braginskii, Epperlein and Haines, and Ji and Held are shown as curves, and direct solutions from (a) the Fokker–Planck code *OSHUN* and (b) Ji and Held's 320-term expansion are shown as squares.

motivated a reconsideration of well-established fits. Braginskii's fits were found to be more accurate than expected, with the only significant error being an overestimate in advection due to perpendicular resistivity at intermediate Hall parameters. It is also worth noting that Braginskii underestimates the Nernst velocity by up to 10% at small Hall parameters. Epperlein and Haines' fits give physically incorrect results for advection due to perpendicular resistivity and perpendicular electrothermal coefficient, greatly overestimating these effects for Hall parameters <1. It is also worth noting that Epperlein and Haines' fits for perpendicular resistivity show significant discontinuities in their variation with Z. Ji and Held's fits give physically incorrect results for advection due to perpendicular resistivity and perpendicular electrothermal coefficient, but the errors are only significant near the zero Hall parameter. Ji and Held's fits are the only ones considered that are continuous functions of Z, valid from 0 to 100. New fits have been obtained,[2] following Ji and Held's approach, that give significantly improved accuracy for magnetic-field advection.

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