# Non-LTE Effects on the Speed of Sound in Plasmas

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**Wen-fai Fong**
Advisor: Reuben Epstein

**Summer High School Research Program 2003**
Laboratory for Laser Energetics
University of Rochester

**Abstract**

Local thermodynamic equilibrium (LTE) has been the standard assumption for thermodynamic calculations of plasma properties. Assuming non-LTE requires a modification of the first law of thermodynamics to account for radiation loss from the plasma. A corresponding correction to the entropy balance is added to complete a thermodynamic model that is consistent with a modification of LTE ionization. Under non-LTE, I consider a simplified collisional radiative equilibrium (CRE) model that accounts for radiative recombination as well as for collisional ionization and collisional recombination in determining the ionization state of a plasma. A sound wave creates perturbed regions of compression and decompression throughout the medium, in which the photon emission rate due to radiative recombination varies, relative to a uniform energy supply used to maintain the unperturbed plasma at a steady state. Using an adiabatic exponent accounting for the ionization energy of the plasma, an expression for the speed of sound is obtained in terms of various thermodynamic quantities. One-tenth solid density aluminum is used as an illustrative example. The speed of sound is plotted versus temperature for both LTE and CRE to show where there is a significant difference between these two assumptions.

# I. Introduction

The propagation of a sound wave through a radiating plasma is both an irreversible and adiabatic process. A sound wave creates areas of compression throughout its medium, thereby causing the pressure of the medium to vary. Due to the rapidity of these pressure variations, we may assume that a negligible amount of thermal energy is exchanged between the compressed regions and their surroundings. Such a process is referred to as an *adiabatic* process.[1] An *irreversible* process is one in which the entropy of the universe permanently increases. Entropy ($S$) is a state function $S(T, V)$ and is expressed in LTE as:

$$dS = \frac{1}{T}dU + \frac{P}{T}dV \qquad (1)$$

where $U$ is the internal energy, $V$ is the volume and $T$ is the temperature. At constant entropy, when the medium is compressed, energy radiates due to radiative recombination.[2]

Local thermodynamic equilibrium (LTE) is the standard assumption for thermodynamic calculations. In assuming LTE, we assume time-reversibility of all processes. Non-local thermodynamic equilibrium (non-LTE) introduces irreversibility. Collisional-radiative equilibrium (CRE) is a case of non-LTE which includes the effects of both radiative and collisional recombination. Radiative recombination is a two-body interaction that occurs when an electron passes nearby a positive ion. The ion's electrostatic forces accelerate the electron, thus causing it to radiate and emit a photon. With the emission of the photon, the electron's velocity decreases, and it becomes bound to the ion. When the photons escape from the plasma, radiative recombination causes a loss of energy. On the other hand, collisional recombination is a three-body interaction, occurring when two free electrons collide near an ion. One electron remains free, while the other becomes bound to a nearby positive ion. Unlike radiative recombination, collisional recombination does not entail energy loss from the plasma.

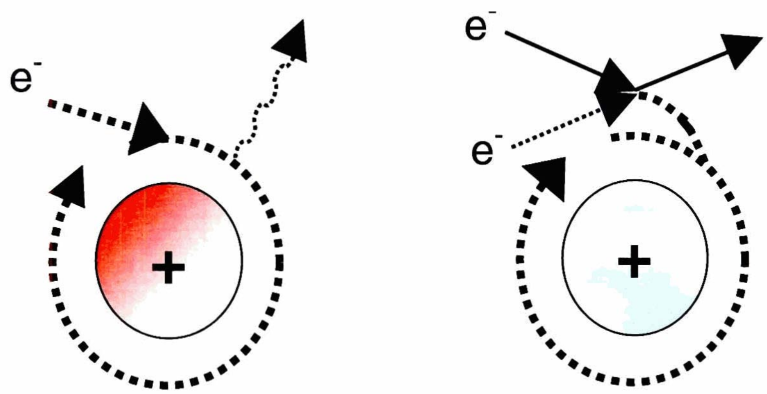


Fig. 1. Radiative recombination (left), and collisional recombination are displayed.

Collisional-radiative equilibrium accounts for radiative recombination as well as collisional ionization and recombination. However, it does not account for photoionization, the time-reversed process of radiative recombination. Therefore, the energy lost in radiative recombination is not recovered by the plasma itself.

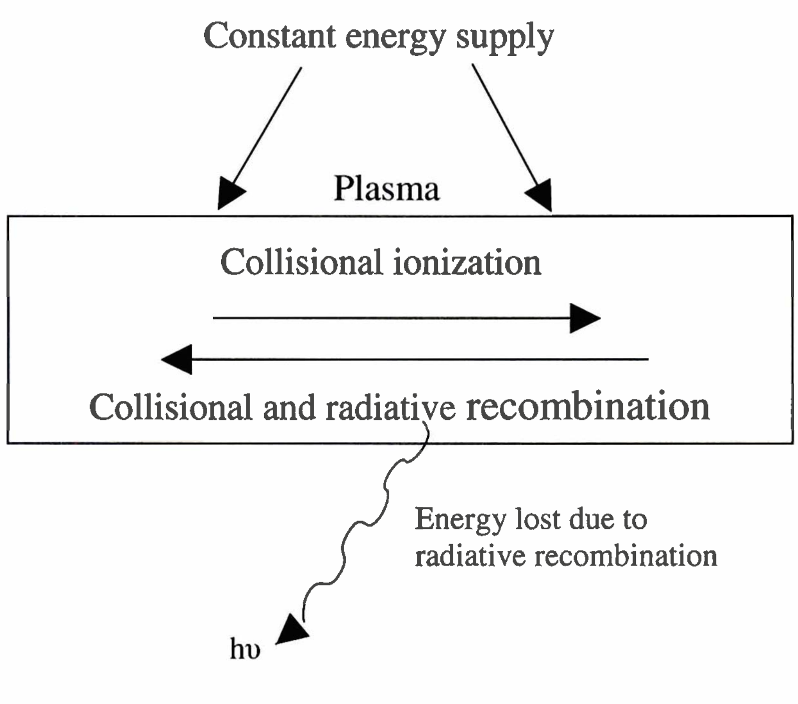


Fig. 2. A conceptual picture of the energy balance in the plasma to maintain a steady state.

As a sound wave propagates through the plasma, it creates regions of compression and decompression throughout the medium. In regions of compression, there is an increased number of collisions as particles are pushed closer together. The radiative recombination rate increases with compression, along with energy loss. In regions of decompression, there is a decreased number of collisions as particles are pushed farther apart, resulting in a decrease in the rate of radiative recombination and less energy loss. A constant, uniform energy supply is provided to the plasma to compensate for the average rate of energy loss due to radiation and thus maintains the overall steady state of the unperturbed plasma. Collisional radiative equilibrium, under non-LTE, accounts for both collisional and radiative recombination, assuming that no photoionization occurs.[3]

# II. Calculations

The exchange of heat $d_e Q$ into the plasma from the environment is given by:

$$d_e Q = dU + pdV \tag{2}$$

where $Q$ is the heat of the plasma, $U$ is the internal energy of the plasma, and $p$ and $V$ are the pressure and volume of the plasma, respectively. Prigogine and Kondepudi[3] account for the interior and exterior energy and entropy balances of the plasma by the first and second laws of thermodynamics. The irreversible radiated energy loss in excess of the energy added by the total constant, uniform heat source is denoted as $d_i Q$. The total change in the heat of the plasma is

$$dQ = d_e Q + d_i Q\,. \tag{3}$$

Assuming CRE, the energy balance is accounted for by

$$dQ = dU + pdV + \mu_{CR} dN^+, \tag{4}$$

where $d_i Q = \mu_{CR} dN$ accounts for the loss of energy due to radiative recombination. Dr. Epstein[4] expresses $\mu_{CR}$ as the ionization energy $\chi(N^+)$ multiplied by the probability that recombinations occur radiatively, as opposed to collisionally. In LTE, $\mu_{CR}$ is equal to zero, so $dQ = dU + pdV$.

A corresponding correction is made to the entropy balance:

$$dS = \frac{1}{T}dU + \frac{P}{T}dV + \frac{\mu_{CR}}{T}dN^{+}. \tag{5}$$

$U$ is a function of temperature and particle numbers and can be defined by

$$U = \frac{3}{2}NkT + \int^{N^+} \chi(N^{+'})dN^{+'} \tag{6}$$

$$dU = \frac{3}{2}NkdT + \left(\frac{3}{2} + \chi\right)dN^{+} \tag{7}$$

where $N^+$ is the number of ions in the plasma in a given ionization state and $k$ is Boltzmann's constant. $\chi(N^+)$ is the ionization energy for a particular ionization stage and is a slowly varying function of $N^+$. Notice that $3/2\ NkT$ is the expression for energy of ideal monatomic gases . The equation of state is

$$PV = NkT, \tag{8}$$

where $N$ is the number of particles in the plasma. Using Eq. 4, the specific heats at constant volume and at constant pressure follow, respectively:

$$c_V = \left(\frac{\partial Q}{\partial T}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V + \mu_{CR}\left(\frac{\partial N}{\partial T}\right)_V \tag{9}$$

$$c_P = \left(\frac{\partial Q}{\partial T}\right)_P = \left(\frac{\partial U}{\partial T}\right)_P + P\left(\frac{\partial V}{\partial T}\right)_P \mu_{CR}\left(\frac{\partial N}{\partial T}\right)_P \tag{10}$$

To simplify these expressions, we introduce the parameters $\eta_V \equiv \frac{V}{N}\left(\frac{\partial N}{\partial T}\right)_V$ and $\eta_T \equiv \frac{T}{N}\left(\frac{\partial N}{\partial V}\right)_T$, where the change in the number of ions $N^+$ equals the change in the number of free particles $N$ ($dN = dN^+$). The specific heats are expressed as:

$$c_V = \frac{3}{2}Nk + Nk\eta_T\left(\frac{3}{2}kT + \frac{\chi + \mu_{CR}}{kT}\right) \tag{11}$$

$$c_P = \frac{5}{2}Nk + \left[\frac{(\eta_T + \eta_V)\left(\frac{5}{2} + \frac{\chi + \mu_{CR}}{kT}\right)(Nk)}{1 - \eta_V}\right]. \tag{12}$$

In Clayton,[5] a dimensionless quantity $\gamma_1$ is the first adiabatic exponent out of three. It is defined as

$$\frac{dp}{p} + \gamma_1 \frac{dV}{V}. \tag{13}$$

Clayton[5] also derives an expression for $\gamma_1$ in terms of the specific heats at constant volume and pressure, respectively:

$$\gamma_1 = -\frac{c_P}{c_V}\left(\frac{\partial p}{\partial V}\right)_T \frac{V}{p} \tag{14}$$

$$\left(\frac{\partial p}{\partial V}\right)_T = \frac{NkT(\eta_V - 1)}{V^2} \tag{15}$$

This is purely a result of the first law of thermodynamics and the equation of state. If Eq. 11 and Eq. 12 are substituted into Eq. 14, $\gamma_1$ is now expressed in terms of ionization:

$$\gamma_1 = \frac{\frac{5}{2}(1+\eta_T) + (\eta_T + \eta_V)\left(\frac{\chi + \mu_{CR}}{kT}\right)}{\frac{3}{2}(1+\eta_T) + \eta_T\left(\frac{\chi + \mu_{CR}}{kT}\right)}. \tag{16}$$

The parameters used are now more specifically defined in terms of ionization:

$$\eta_T = \frac{T}{1+Z}\left(\frac{\partial Z}{\partial T}\right)_V \tag{17}$$

$$\eta_V = -\frac{\rho}{1+Z}\left(\frac{\partial Z}{\partial \rho}\right)_T . \tag{18}$$

Entropy $S$ (Eq. 1) is an analytic function and therefore displays reciprocity:

$$\frac{\partial}{\partial T}\left[\left(\frac{\partial S}{\partial V}\right)_T\right]_V = \frac{\partial}{\partial V}\left[\left(\frac{\partial S}{\partial T}\right)_V\right]_T . \tag{19}$$

This characteristic indicates that the value of entropy does not depend on its history, but rather, on its present state. $S$ can now be expressed in terms of the parameters just defined:

$$\eta_V\left(\frac{3}{2}kT + \chi + \mu_{CR}\right) = \eta_T kT . \tag{20}$$

Eq. 20 can be used to simplify Eq. 16.

$$\gamma_1 = \frac{\frac{5}{2}\eta_V(1+\eta_T) + (\eta_T+\eta_V)\left(\eta_T - \frac{3}{2}\eta_V\right)}{\frac{3}{2}\eta_V(1+\eta_V) + \eta_T\left(\eta_T - \frac{3}{2}\eta_V\right)} . \tag{21}$$

Assuming that this propagation is an adiabatic change with radiation energy loss and both the energy and entropy balances accounted for,

$$c_S{}^2 = \frac{\gamma_1 p}{\rho} . \tag{22}$$

Eq. 16 and Eq. 22 give

$$c_S{}^2 = \frac{\frac{5}{2}(1+\eta_T) + (\eta_T+\eta_V)\left(\frac{\chi+\mu_{CR}}{kT}\right)}{\frac{3}{2}(1+\eta_T) + \eta_T\left(\frac{\chi+\mu_{CR}}{kT}\right)}\frac{NkT}{V\rho} , \tag{23}$$

where $\rho$ is the mass density. Plugging Eq. 20 into Eq. 23,

$$c_S{}^2 = \frac{\frac{5}{2}\eta_V(1+\eta_T)+(\eta_V+\eta_T)\left(\eta_T-\frac{3}{2}\eta_V\right)}{\frac{3}{2}\eta_V(1+\eta_T)+\eta_T(\eta_T-\frac{3}{2}\eta_V)}\frac{NkT}{V\rho} \tag{24}$$

is obtained.

This expression for the speed of sound applies to both LTE and CRE. The Saha equation for LTE is stated

$$\frac{N^{n+1}N_e}{N^n} = AT^{\frac{3}{2}}e^{-\frac{\chi}{kT}} \tag{25}$$

where $A$ is a constant. For a partially ionized gas, $N^n$ is the number of n-times ionized ions in a given ionization state and $N_e$ is the number of electrons.[6] Assuming CRE, the Saha equation can be modified in order to be consistent with the thermodynamic consistency condition Eq. 20,

$$\frac{N^{n+1}N_e}{N^n} = AT^{\frac{3}{2}}e^{-\frac{\chi+\mu_{CR}}{kT}} \tag{26}$$

Eq. 21 can now be evaluated in a thermodynamically consistent manner in terms of Eq. 26.

# III. Results

For evaluating Eq. 24, tables of $\chi$, $\frac{T}{Z}\left(\frac{\partial Z}{\partial T}\right)_V$, $\frac{\rho}{Z}\left(\frac{\partial Z}{\partial \rho}\right)_T$, and average Z for aluminum plasma at one-tenth solid density were provided. For CRE, all ionization effects of spontaneous emission were removed. These were obtained from collisional-radiative calculations provided by Dr. R. Epstein.[4]

The average Z is plotted in Fig. 3 against temperature. The CRE curve displays a similar pattern to the LTE curve, but requires a higher temperature to reach a given ionization state. This is due to the lack of photoionization occurring in a non-LTE plasma, where energy is lost due to radiative recombination. A "jump" occurs at eleven detached electrons (z=11)

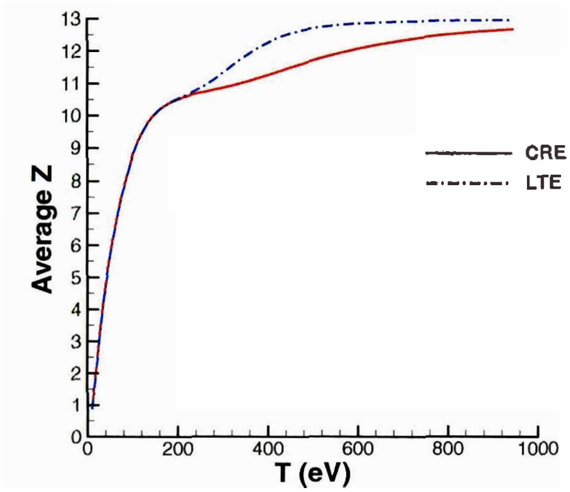


Fig. 3. The average charge per ion for aluminum at one-tenth solid density is plotted as a function of temperature.

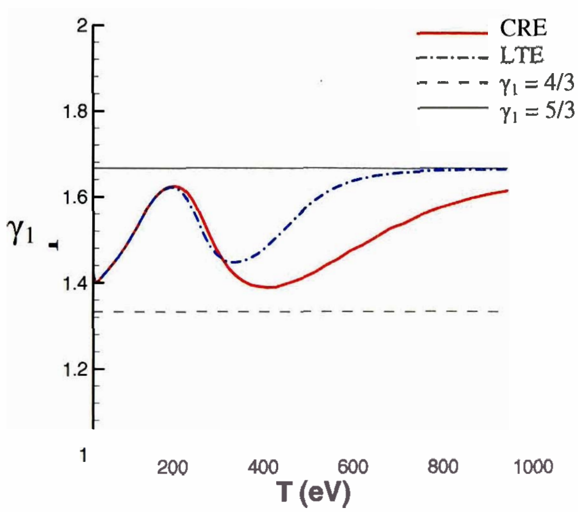


Fig. 4. $\gamma_1$ for aluminum at one-tenth solid density is plotted as a function of temperature.

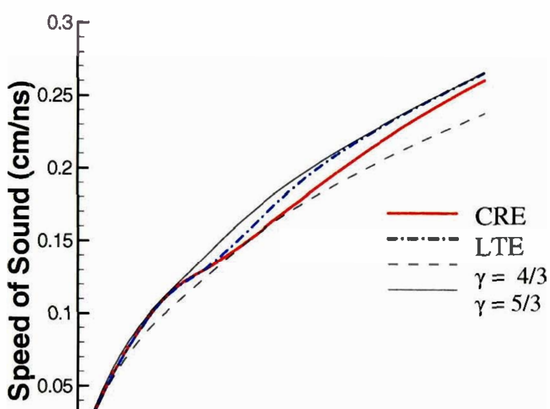


Fig. 5. The speed of sound for aluminum at one-tenth solid density plotted as a function of temperature.

because only the innermost complete shell is left, and a significant increase in the ionization energy occurs at that point.

A Fortran program[7] is used to evaluate the values of $\gamma_1$ and the speed of sound for CRE and LTE through Eq. 21 and 24. These values are then plotted against temperature (Fig. 4). Like Fig. 2, the CRE curve displays a similar pattern to the LTE curve but with a similar temperature shift. In this case, $\gamma_1 = 5/3$ corresponds to a neutral gas in which no ionization occurs, or a fully ionized plasma or a plasma with constant ionization.

Since $\gamma_1$ is an important part of the speed of sound equation, the same pattern can be seen in Fig. 5, where the speed of sound is plotted. Once again, the CRE curve displays a similar pattern to the LTE curve.

## IV. Significance

The assumption of non-LTE thermodynamics is significant in that the behavior of sound waves propagating through a plasma is similar to that of shock waves, which often propagate within the target chamber. Studying the apparent difference between the speed of sound under both LTE and non-LTE conditions can further laser-induced nuclear fusion studies. These differences can be attributed to ionization effects. These non-LTE effects should be studied in other low-density substances, in order to find other cases of interest. For instance, carbon at one-thousandth solid density is another interesting case where there is a significant difference in average Z between LTE and CRE. There also may be other ways to perform the non-LTE calculations, taking factors other than ionization, energy and entropy into account.

## V. Conclusions

In order to perform thermodynamically consistent calculations of the speed of sound of an aluminum plasma at one-tenth solid density, the energy and entropy balance must be accounted for in the first and second laws of thermodynamics. This effect of radiative recombination can be accounted for by the $\mu_{CR}$ term in the energy balance. This allows thermodynamic consistency in the calculation for the $\gamma_1$ and the speed of sound. As shown by the case of aluminum at one-tenth solid density, the effects of non-LTE and CRE can be significant at low densities. Given a high-density substance, collisional ionization and recombination are favored over radiative recombination and photoionization. For low-density substances such as gases and plasmas, the effects of non-LTE and CRE can be seen because the radiative processes are more important, and the energy lost due to radiation is accounted for. It can be reasonably conjectured that significant differences in LTE and non-LTE are likely to be seen where these effects are seen in ionization.

## VI. Acknowledgments

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