Design Methods for Two Regimes of Unobscured Reflective Optical Systems

by

Eric M. Schiesser

Submitted in Partial Fulfillment of the

Requirements for the Degree

Doctor of Philosophy

Supervised by Professor Jannick P. Rolland

The Institute of Optics

Arts, Sciences and Engineering

Edmund A. Hajim School of Engineering and Applied Sciences

University of Rochester

Rochester, New York

2019
Dedication

This work is dedicated to my fiancé Sarah, who has motivated and supported me in my studies and beyond; to my parents, my mother Kimberlee, my father Mark, and Bridgette for providing the many avenues of support I needed; and to my grandfather David for inspiring me to study optics.
# Table of Contents

Biographical Sketch ................................................................. vi

Acknowledgments ................................................................. viii

Abstract ................................................................................... x

Contributors and Funding Sources .......................................... xi

List of Tables ............................................................................ xii

List of Figures .......................................................................... xiii

List of Symbols and Abbreviations .......................................... xvii

1  Introduction ............................................................................. 1

   1.1  Background ......................................................................... 2

   1.2  Aberration theory in unobscured mirror systems ....................... 6

   1.3  Two unobscured optical design regimes .................................. 10

   1.4  Dissertation Outline ........................................................ 12

2  Unobscured four-mirror laser relay ...................................... 14

   2.1  Introduction to the AIR .................................................... 14

   2.2  First-order optics ............................................................ 16

      2.2.1  First-order 4-mirror solution ....................................... 17

   2.3  Dual-Cassegrain and Gaussian Curvature ............................ 20

      2.3.1  Coddington's Equations and Gaussian curvature ................. 23

      2.3.2  Finding the coaxial curvature of a hyperboloid from the off-axis parameters .................................................. 26
2.3.3 Application to the design process ............................................................ 29
2.3.4 Conclusion for Section 2.3 ...................................................................... 33
2.4 Unobscured tilted-component telescope for AIR ........................................... 33
  2.4.1 Space constraints .................................................................................. 34
  2.4.2 Unobscured tilted component telescopes .............................................. 37
  2.4.3 AIR Alignment Test-bed ......................................................................... 40
  2.4.4 Course Alignment .................................................................................. 42
  2.4.5 Fine Alignment ..................................................................................... 43
  2.4.6 Alignment results .................................................................................. 46
  2.4.7 Magnification measurement .................................................................. 50
  2.4.8 Conclusion to Section 2.4 ..................................................................... 53
3 NAT-based ratio method of freeform design .................................................... 54
  3.1 Introduction ............................................................................................. 54
  3.2 Expansion of NAT astigmatism to 8th order .............................................. 55
    3.2.1 Field dependence of Zernike Astigmatism ($Z_5$ & $Z_6$) up to 8th order .. 57
  3.3 Fitting the $Z_{5/6}$ FFD with NAT field dependence .................................... 65
    3.3.1 Plane symmetric optical systems ......................................................... 67
  3.4 Estimating required surface departure ....................................................... 70
    3.4.1 Zernike Astigmatism shape ($Z_5$) ......................................................... 71
    3.4.2 Zernike Coma shape ($Z_8$) ................................................................. 72
    3.4.3 Zernike Trefoil shape ($Z_{11}$) ............................................................... 74
  3.5 Design Example ....................................................................................... 75
3.5.1 Correcting $B_{222}^2$ .............................................................. 77
3.5.2 Correcting $A_{222}$ and $A_{131}$ simultaneously ................................. 79
3.5.3 Correcting $C_{331}^3$ and $C_{422}^3$ simultaneously with a Z11 shape on two mirrors ....................................................................................................................... 80
3.5.4 Discussion .................................................................................. 81
3.5.5 Higher order Zernike surface coefficients ........................................ 82
3.6 Conclusion ...................................................................................... 83
3.7 Chapter 3 Appendix ........................................................................ 83
4 Volume comparison of unobscured TMC designs .................................... 85
4.1 Traditional TMC design – the TMCA ............................................... 88
4.2 Conversion of TMCA designs to centered Zernike surfaces ............... 93
4.3 Additional WFE correction for the TMCF Converted designs using freeform surfaces ................................................................................................................. 94
4.4 Additional volume reduction using freeform surfaces......................... 95
4.5 Discussion ...................................................................................... 96
4.5.1 Iso-volume comparison ................................................................. 96
4.5.2 Iso-performance comparison ......................................................... 101
4.6 Conclusion to Section 4 .................................................................. 103
4.7 Chapter 4 Appendix ......................................................................... 105
5 Conclusion and Future Work .................................................................. 108
References ............................................................................................ 110
Biographical Sketch

Eric M. Schiesser was born in Rochester, NY and grew up in the neighboring town of Webster. In the Spring of 2009, he began studying Physics and Optics at the University of Rochester as an undergraduate. In the intervening summers, he enjoyed filling his time with internship positions all over the country: Stanford Research Institute in Menlo Park, CA; Texas Instruments in Plano, TX; and Corning Incorporated in Corning, NY. In 2012 he graduated *magna cum laude* with a Bachelor of Science degree in Physics and a Bachelor of Science degree in Optical Engineering. He enrolled in the doctoral program at The Institute of Optics in the Fall of 2012. During his tenure, he was awarded the Frank J. Horton Research Fellowship from 2014-2019 and the Michael Kidger Memorial Scholarship in Optical Design in 2015. He pursued research in optical design and aberration theory under the direction of Professor Jannick P. Rolland.

The following peer-reviewed publications were a result of work conducted during doctoral study:

**Journal Articles**


**Conference proceedings**

*Speaker*


*Contributing author*


Acknowledgments

I would first like to thank my advisor, Professor Jannick P. Rolland, for guiding me through my academic studies. Her strong leadership often pushed me to seek answers to previously unforeseen questions. Additionally, her fierce support of her students helped me gain the confidence to share my ideas.

I would like to thank Dr. Seung-Whan Bahk for his endless support in the lab. Dr. Bahk gave me much of his time when I needed help making something work. He also created the wavefront sensor and much of the associated software that I used. His all-spherical achromatic image relay concept was the basis for much of my work at LLE. His help and insight were critical to my success.

I would like to thank Dr. Jake Bromage for his guidance and encouragement.

Thank you to Michael Spilatro and Dr. Ben Webb for their lab assistance.

I would like to give special thanks Dr. Aaron Bauer. Aaron was not only like a second advisor, but also a great friend in the office. I will miss our daily lunch hour and Bills talk.

I would like to thank some of my officemates, especially Jacob Reimers, Jonathan Papa, Nick Takaki, Di Xu, Romita Chaudhuri, Changsik Yoon, and Yuxuan Liu for participating, freely or by proximity, in what eventually came to be known as “the Distraction Zone.” Additionally, thanks to Dr. Robert Gray, Dr. Kyle Fuerschbach, Dr. Cristina Canavesi, Dr. Jianing Yao, and Daniel Nikolov for at some point sharing an office and contributing to discussions therein.
I would like to add a special word about the late Dr. Kevin Thompson. There are many stories out there about the special person Kevin was in the lives of everyone he worked with, and mine is no exception. Kevin always had kind, reassuring insight and a generosity of spirit that I looked forward to when asking for his help. He was a wonderful mentor in our short time together and I wish I could share this work with him as he shared his with me.

And finally, thank you to my thesis committee members, Professor Miguel Alonso, Professor Jake Bromage, and Dr. Seung-Whan Bahk, and my committee chair, Professor Ronni Pavan, for serving on my dissertation committee.
Abstract

Unobscured reflective optical systems require unique optical design techniques due to their lack of rotational symmetry. The aberrations introduced to avoid obscuration can be understood in the context of Nodal Aberration Theory (NAT), which describes the way optical aberrations add when components are tilted and decentered relative to their axes of symmetry. In this work, we will explore the design methods for two types of unobscured reflective optical systems: a high-power laser image relay and a wide field-of-view telescope. The requirements of each of these optical systems place them in different design regimes, each requiring unique design techniques to achieve adequate performance. The laser image relay, which is designed for an upgrade to the Multi-terawatt Optical Parametric Amplifier Line (MTW-OPAL) at the Laboratory for Laser Energetics (LLE), can be achieved with traditional off-axis conics or spherical mirrors. Two design approaches for this relay will be explored: an axi-symmetric series of off-axis conics and a series of tilted spherical mirrors. On the other hand, the wide field-of-view telescope requires the use of freeform optics for adequate correction. To aid in its design, we extend the NAT astigmatism terms to 8th order to ascertain the higher-order field dependence of the Zernike astigmatism aberrations, and then use this field-dependence to estimate the surface coefficients required to simultaneously correct certain low-order aberrations according the aberration theory of freeform surfaces (ATFS). Finally, we show the quantitative effect of freeform surfaces on the volume and performance of a three-mirror telescope design type called the three-mirror compact (TMC).
Contributors and Funding Sources

This work was supported by a dissertation committee consisting of Professors Jannick P. Rolland (advisor) and Miguel Alonso of The Institute of Optics, and Professor Jake Bromage and Dr. Seung-Whan Bahk of the Laboratory for Laser Energetics. The all-spherical achromatic image relay design form analyzed in Section 2.4 was provided by Dr. Seung-Whan Bahk. All other work was completed independently by the student. Graduate study was supported, in part, by a Frank J. Horton Graduate Research Fellowship from the University of Rochester’s Laboratory for Laser Energetics (DE-NA0001944). Additional support was provided by the National Science Foundation I/UCRC Center for Freeform Optics (IIP-1338877, IIP-1338898, IIP-1822049 and IIP-1822026).
List of Tables

Table 1.1: Select specifications for the WFOV freeform imager and Achromatic Image Relay for comparison......................................................................................................................12

Table 2.1. Required off-axis parameters for the long and short image relays used to compute the axial parameters given in Table 2.2. .................................................................32

Table 2.2. Computed axial parameters corresponding to the required off-axis parameters in Table 2.1. These parameters describe the designs in Figs. 2.4 and 2.5.................................................................................................................................32

Table 2.3. The input and output values of the paraxial equations for the AIR. The values $d_0, d_1, d_2, d_3, d_4, c_1$ are the inputs and $c_2, c_3, c_4$ are the output values. ...................36

Table 2.4. Angles of AIR mirrors for three designs: the two solutions to Eqs. (2.36) and (2.37) ........................................................................................................39

Table 2.5. Description of each fine alignment step, including the amount that the given actuator was adjusted and the amount that is predicted for that actuator ..........50

Table 3.1 Field dependence of Z5 and Z6 for a y-z plane symmetric system, broken down into each NAT term and listed in ascending wavefront expansion order ...69

Table 3.2. B222 removal process with two iterations..........................................................78

Table 3.3. A222/A131 ratio for each pair of surfaces and the resulting predicted surface coefficients .............................................................................................................80

Table 4.1. Naming convention and descriptions for the designs completed in this work ........................................................................................................................................88

Table 4.2. System specifications for the TMCA and TMCF designs ................................90

Table 4.3. Departure from Base Sphere for each design type in the iso-volume comparison .........................................................................................................................101

Table 4.4. Departure from base sphere for each design type in the iso-performance comparison ..................................................................................................................103

Table 4.5. The FRINGE Zernike coefficients for the primary mirror representing departure from base sphere for each diffraction-limited design. .........................105

Table 4.6. The FRINGE Zernike coefficients for the secondary mirror representing departure from base sphere for each diffraction-limited design. .........................106

Table 4.7. The FRINGE Zernike coefficients for the tertiary mirror representing departure from base sphere for each diffraction-limited design. .........................107
List of Figures

Figure 1.1 Aberrations for a spherical mirror with the aperture stop at the center of curvature for the on-axis case, on-axis case with a larger aperture, and off-axis case..........................................................5

Figure 1.2. The entrance pupils of the spherical mirror configurations seen in Figure 1.1. The shaded area indicates the aperture area relative to the full 300 mm entrance pupil.................................................................6

Figure 1.3 Illustration of sigma vector geometry in NAT. OAR: optical axis ray. CoC: center of curvature. XP: exit pupil. Imaging Axis: object-image axis of symmetry. Pupil Axis: pupil axis of symmetry. N: surface normal.........................7

Figure 1.4 Field coordinate and pupil coordinate vectors....................................................8

Figure 2.1 Layout of a back-to-back Cassegrain relay with an off-axis chief ray traced through the system. The intersection points of the ray with each nth mirror are labeled \( P_n \). The off-axis distances are labeled \( d_n \). The vertices are labeled \( V_n \), and the stigmatic imaging points of each mirror are labeled \( S_n \). The stigmatic imaging points overlap for each successive mirror to stigmatically relay between collimated object space to collimated image space........................................22

Figure 2.2 Geometry of a hyperboloid used as a Cartesian reflector. The incoming ray (solid red) is directed toward the point \( F \), the front focal point of the hyperboloid. The point \( A \) is the point of intersection of the ray and the surface. The dashed blue line is the surface normal at \( A \). \( C \) is the geometric center of the hyperboloid. \( R_s \) is the local sagittal RoC..............................................................26

Figure 2.3 Grating compressor chamber (GCC) layout showing locations of N5 crystal plane and G4 grating plane. The available space for the final design of the achromatic image relay (AIR) are shown. All optics between N5 and the AIR are flat mirrors. All optics between the AIR and G4 are flat optics with regard to imaging. .............................................................29

Figure 2.4 Layout of the "long" relay, with paraboloids in green and hyperboloids in red. See Table 2.1 for off-axis requirements and Table 2.2 for the computed axial values.................................................................30

Figure 2.5 Layout of the "short" relay, with paraboloids in green and hyperboloids in red. Table 2.1 for off-axis requirements and Table 2.2 for the computed axial values. .................................................................30

Figure 2.6. (a) The contents of the GCC. The AIR mirrors are labelled M1, M2, M3, and M4 and the AIR is outlined and shaded in red. The grating compressor is outlined in blue. The final grating of the grating compressor is labelled G4.
The input plane for the AIR is the NOPA5 crystal, labelled N5. The chief ray of the beam path is shown in red. (b) The GCC with everything except the AIR removed, with the beam path still shown. (c) The test-bed setup to test the alignment of the AIR. The pinhole jig locations are labeled as Pn. The chief ray of the beam path is shown in red.

Figure 2.7. Illustration of the minimum beam size requirement on M2 and M3 given by Eq. (2.33). \( f_1 \) is the focal length of M1. \( D_1 \) and \( D_2 \) are the beam diameters on M1 and M2, respectively. \( d \) is the signed distance between M1 and M2.

Figure 2.8. Layout of solution 1 from Table 2.4.

Figure 2.9. FFDs for solution 1.

Figure 2.10. Layout of solution 2 from Table 2.4.

Figure 2.11. FFDs of solution 2.

Figure 2.12. The test-bed components to produce a 45 x 45 mm beam and to measure the return wavefront. WFS – Wavefront sensor. Cam – Camera. QWP – quarter wave plate.

Figure 2.13. Adjustments for M3 and M4. Red arrows indicate actuator adjustments. Blue arrows indicate the motion of the mirror.

Figure 2.14. Algorithm for adjusting a single actuator to minimize a given Zernike mode.

Figure 2.15. The double pass wavefront given in waves at 532 nm for (a) after the course alignment step and (b) after the fine alignment steps. Note that the color scale ranges from \([0,2]\) waves for (a) and from \([0,1]\) waves for (b).

Figure 2.16. The FRINGE Zernike mode coefficient values at each step for terms Z4 to Z9 given in waves at 532 nm.

Figure 2.17. The peak-to-valley and root-mean-square (rms) values of the wavefront at each step.

Figure 2.18. The rms wavefront error values of the wavefront at each step at a smaller scale to show detail.

Figure 2.19. Direction of motion of M3 (indicated by the red arrow) to compensate magnification error.

Figure 3.1 The Zernike Z5/6 Full Field display of each NAT plane-symmetric field-dependence.

Figure 3.2. The layout of the example TMC design from Bauer et al. with freeform surface terms removed, leaving only the base spherical surfaces.

Figure 3.3. Full field displays for the design with all spherical surfaces. Units are waves at 532 nm. (a) RMS WFE (b) Defocus (Z4) (c) Astigmatism, Z5/6 (d) Coma, Z7/8 (e) Spherical, Z9 (f) Elliptical Coma, Z10/11.
Figure 3.4. Selected NAT coefficients related to the Zernike surface terms Z5 (astigmatism), Z8 (coma), and Z11 (trefoil, or elliptical coma) shown at each step.

Figure 3.5. (a) The ratios of the $A_{222}$ and $A_{131}$ from adding a Z8 shape onto each mirror and for the overall aberrations seen in Figure 3.4. (b) The ratios of the $C_{422}$ and $C_{333}$ from adding a Z11 shape onto each mirror and for the overall aberrations seen in Figure 3.4.

Figure 3.6. Astigmatism FFDs after each step. The axes units are the object field angle coordinates in degrees. (a) Starting design without any freeform (spheres only). (b) After $B_{222}$ removal. (c) After $A_{222}/A_{131}$ removal. (d) After $C_{422}/C_{333}$ removal. (e) After removing the residuals of each terms through another iteration of each step.

Figure 3.7. Zernike Coma (Z7/8) FFDs for each step. The axes units are the object field angle coordinates in degrees. (a) Starting design without any freeform (spheres only). (b) After $B_{222}$ removal. (c) After $A_{222}/A_{131}$ removal. (d) After $C_{422}/C_{333}$ removal. (e) After removing the residuals through another iteration of each step.

Figure 3.8. Zernike Elliptical Coma (Z10/11) for each step. The axes units are the object field angle coordinates in degrees. (a) Starting design without any freeform (spheres only). (b) After $B_{222}$ removal. (c) After $A_{222}/A_{131}$ removal. (d) After $C_{422}/C_{333}$ removal. (e) After removing the residuals through another iteration of each step.

Figure 4.1. The layout of the threemrc.len example lens in CODE V, a TMCA type design.

Figure 4.2. Diagram of surface-ray intersection coordinates (black dots) in the Y-Z plane at x=0 and three bounding boxes in different coordinate frames (red, blue and green).

Figure 4.3. (a) Field-averaged RMS WFE performance versus volume for each design type. The original TMCA design optimized for volume (blue line) crosses the 0.07 waves line at 96.9 L. The TMCF Frozen-Geometry design (red line) crosses the 0.07 waves line at 72.6 L. The TMCF Volume-Optimized design (yellow line) crosses the 0.07 wave line at 59.0 L. The reverse optimization from 59 L to 110 L of the TMCF Volume-Optimized design (dashed green line) avoids the local minimum of the TMCF Volume-Optimized forward optimization. (b) The same data as the chart in (a), showing more detail in the 0.0 to 0.1 waves range.

Figure 4.4. Layouts of each design type corresponding to a volume of 72.5 L, the smallest diffraction-limited volume of the TMCF Frozen-Geometry: (a) The TMCA, (b) TMCF Frozen-Geometry, and (c) the TMCF Volume-Optimized designs. Note that the apparent overlap of the surfaces in the layouts is due to the extension of the surfaces in the drawing program. The clear apertures and the rays have no conflicts in the designs.
Figure 4.5. Zernike defocus (Z4) FFD for the (a) TMCA design, (b) TMCF Frozen-Geometry design, and (c) TMCF Volume-Optimized design. Blue indicates a positive value, red indicates a negative value. .......................................................98

Figure 4.6. Zernike Astigmatism (Z5/Z6) FFD for the (a) TMCA design, (b) TMCF Frozen-Geometry design, and (c) TMCF Volume-Optimized design. .................................98

Figure 4.7. Zernike coma (Z7/Z8) FFD for the (a) TMCA design, (b) TMCF Frozen-Geometry design, and (c) TMCF Volume-Optimized design. ..............................99

Figure 4.8. Zernike spherical aberration (Z9) FFD for the (a) TMCA design, (b) TMCF Frozen-Geometry design, and (c) TMCF Volume-Optimized design. Blue indicates a positive value, red indicates a negative value. ..............................................99

Figure 4.9. Zernike elliptical coma (Z10/Z11) FFD for the (a) TMCA design, (b) TMCF Frozen-Geometry design, and (c) TMCF Volume-Optimized design. ......99

Figure 4.10. Layouts of the smallest volume diffraction-limited designs of each TMC type with their bounding boxes: (a) The TMCA is shown with a blue bounding box, (b) the TMCF Frozen-Geometry is shown with a red bounding box, and (c) the TMCF Volume-Optimized is shown with a green bounding box. (d) The bounding boxes are shown next to each other for perspective. ...........................101

Figure 4.11. Zernike defocus (Z4) FFD for the: (a) TMCA design, (b) TMCF Frozen-Geometry design, and (c) TMCF Volume-Optimized design. Blue indicates a positive value, red indicates a negative value. .....................................................102

Figure 4.12. Zernike Astigmatism (Z5/Z6) FFD for the: (a) TMCA design, (b) TMCF Frozen-Geometry design, and (c) TMCF Volume-Optimized design. ................102

Figure 4.13. Zernike coma (Z7/Z8) FFD for the: (a) TMCA design, (b) TMCF Frozen-Geometry design, and (c) TMCF Volume-Optimized design. .....................102
List of Symbols and Abbreviations

FOV  Field of View
RoC  Radius of Curvature
CoC  Center of Curvature
NF   Near-Field
FF   Far-Field
GC   Gaussian Curvature
TCT  Tilted Component Telescope
AIR  Achromatic Image Relay
LLE  Laboratory for Laser Energetics
ATFS Aberration Theory of Freeform Surfaces
1 Introduction

As optical systems become more specialized, layout and packaging requirements become increasingly strict. Earth-imaging telescopes for small and medium sized satellites (e.g. CubeSats), head-mounted displays for virtual or augmented reality, high-power laser optics, and many other applications typically require not only compact package sizes, but also fully-unobscured forms to achieve high throughput, to reduce diffractive effects, and to avoid optical damage [1–7].

Many of these applications benefit from avoiding refractive surfaces in favor of reflective surfaces, which have myriad advantageous properties. Reflective optical designs can achieve wider spectral bands due to their lack of dispersion. Reflective designs are generally less massive compared to their refractive counterparts. Furthermore, reflective optics can be mounted from the rear using the entirety of the aperture and may therefore be more mechanically stable and could even be actively cooled [8,9].

Reflective optical systems that require no obscuration must operate off-axis in some way. Traditionally, field-bias and aperture offset are used to create an unobscured design using an off-axis portion of a rotationally symmetric parent optic. Often the packaging constraints make it difficult or impossible to create an unobscured design from a rotationally symmetric system using the traditional methods. Furthermore, the restriction to rotationally symmetric parent surfaces reduces the degrees of freedom in the layout of the design, making the packaging requirements even more difficult to achieve. Therefore,
to satisfy the layout and packaging constraints while avoiding obscuration, fully breaking symmetry by tilting the optical components is often required. Tilting the optical components degrades the performance compared to the equivalent (obscured) symmetric design. This degradation is mitigated by two methods: the first is by tilting the mirrors appropriately to partially nullify each other’s aberrations, and the second is by employing a rotationally non-symmetric surface, also known as a “freeform” surface, to correct the aberrations.

This work focuses on two different requirement regimes which may or may not require freeform surfaces for sufficient correction. The goal of this research is to develop methods for designing unobscured optical systems across two specific regimes: small-FOV and high f-number laser beam relays, such as the Achromatic Image Relay (AIR) employed at the Laboratory for Laser Energetics (LLE), and large aperture, wide-FOV freeform telescope systems.

1.1 Background

Reflective optics have been widely used for imaging since the invention of the Newtonian telescope, built by Isaac Newton in 1668. Newton recognized reflective optics lack dispersion, unlike their refractive counterparts. Since the angle of reflection does not depend on wavelength, no dispersion occurs for a reflective optic. This is the main advantage of reflective optical systems: they permit a broad spectral range due to their lack of dispersion.
However, another aspect of reflective systems recognized by Newton is the problem of obscuration. Newton used a small 45-degree fold mirror in his telescope close to the image plane to make the image accessible, but this fold mirror obscures the beam by blocking part of the aperture. Obscuration reduces the signal by reducing the amount of light that makes it to the image plane and can also have undesired diffractive effects due to the support objects. With the advent of more complex reflective systems involving more mirrors, avoiding obscuration becomes challenging and must be carefully considered.

Optical mirror systems without obscuration must break symmetry in some way. Traditionally, this is done by offsetting the aperture, biasing the field, or both simultaneously in an otherwise rotationally symmetric system. In this case, the symmetry of the used optical system no longer maintains the symmetry of its parent, and the used apertures of non-spherical surfaces become non-symmetric sections of their rotationally symmetric parent surfaces.

To illustrate how an offset in aperture changes the aberrations of an otherwise rotationally symmetric optical design, we can look at the simple case of a single spherical mirror with the aperture stop located at the center of curvature (CoC). Figure 1.1 shows the case of a single spherical mirror with the aperture stop at the center of curvature for three different configurations: on axis with a 100 mm entrance pupil diameter (EPD), on-axis with a 300 mm EPD, and off-axis with a 100 mm EPD. In the off-axis case, the aperture stop is decentered by 100 mm such that the upper edge is 150 mm from the central axis. We can see that once the aperture stop is offset, the aberrations no longer share the
symmetry of the parent. We see that the magnitude of the astigmatism and coma aberrations change from rotationally symmetric about the central field to constant over the field.
Figure 1.1 Aberrations for a spherical mirror with the aperture stop at the center of curvature for the on-axis case, on-axis case with a larger aperture, and off-axis case.
Some designs abandon the rotational symmetry of the parent system entirely and tilt the mirror components themselves to avoid obscuration. The tilting of the mirrors and subsequent lack of symmetry introduces aberrations. These aberrations due to tilting components can be dealt with in different ways. Using only spheres, the tilts and radii can be adjusted together to reduce the aberrations in the center of the field-of-view \cite{10}. This typically requires a small field-of-view and/or aperture. For systems with a wider field-of-view or aperture, another degree of freedom to employ is the shapes of the mirrors. In general, the aberrations cannot be mitigated by rotationally symmetric components alone. Freeform optical surfaces, surfaces without rotational symmetry and without an otherwise rotationally symmetric parent, have the potential to address the aberrations caused by the tilting of optical surfaces in wider field systems. To determine which paradigm is appropriate, we require an aberration theory that can account for the lack of symmetry in unobscured systems.

1.2 Aberration theory in unobscured mirror systems

There are multiple methods to construct an aberration theory to handle a lack of rotational symmetry \cite{11–13}. Thompson’s Nodal Aberration Theory (NAT) was originally used to explore the effect of small tilts and decenters on designs that use...
rotationally symmetric components [13]. However, it has recently been applied to the design of rotationally non-symmetric mirror systems [3,14–16]. In addition, Fuerschbach et al. used NAT to determine the field-dependent effects of FRINGE Zernike freeform surfaces on NAT aberrations [17]. NAT can be used to analyze both types of unobscured designs in this work.

The basic premise of NAT is that each rotationally symmetric surface has an axis of symmetry that defines the center of its aberration field, about which the aberrations are the “normal” field-symmetric aberrations by H. H. Hopkins [18]. The tilt and decenter of the surface determine the location of the aberration center relative to the aberration centers of the other surfaces and to the center of the field-of-view. The ray that goes from the center of the FOV through the center of the pupil and through the system to the image plane defines the “optical axis ray” (OAR) and the aberration centers are measured as vectors, called “sigma” vectors, from that ray.

![Figure 1.3 Illustration of sigma vector geometry in NAT. OAR: optical axis ray. CoC: center of curvature. XP: exit pupil. Imaging Axis: object-image axis of symmetry. Pupil Axis: pupil axis of symmetry. N: surface normal.](image-url)
The geometry of the sigma vector can be seen in Figure 1.3. The OAR extends from the central object point to the spherical optical surface. The surface normal extends out from the intersection point through the center of curvature (CoC). The OAR is reflected about this line giving OAR’. Paraxially, the image of the object point can be found by finding the point of intersection of a line from the object point through the CoC with the OAR’. A similar construction joins the CoC with the exit pupil (XP) and entrance pupil along the pupil axis of symmetry line. The sigma vector extends from the image point to the pupil axis line along the paraxial image plane. The cascade of reflections can be traced through a system with multiple mirrors to determine the individual sigma vectors for each surface. Buchroeder presented a paraxial ray-trace method to calculate the individual sigma vectors for each surface, and later Thompson et al. developed a real-ray based method to do the same [19,20].

![Figure 1.4 Field coordinate and pupil coordinate vectors](image)

The key insight of NAT is how the aberrations from different surfaces add together: the Hopkins wavefront is reconstructed using vectors describing the field, pupil, and sigma.
vectors. Equation (1.1) below shows the wavefront aberration summation of NAT. The wavefront aberrations of these systems contain no “new” aberrations but instead result in different field dependence due to the sigma vectors. Therefore, using NAT, it is possible in principle to analyze the aberrations of a system of rotationally symmetric albeit tilted mirrors in terms of the un-tilted paraxial wavefront aberration coefficients of H. H. Hopkins by computing the sigma vectors of the resulting tilted system.

\[
W = \sum_{j}^{N} \sum_{p}^{\infty} \sum_{n}^{\infty} \sum_{m}^{\infty} \left[ W_{k/m,j} \right] \left[ (\tilde{H} - \tilde{\sigma}_j) \cdot (\tilde{H} - \tilde{\sigma}_j) \right]^{p} \left[ \tilde{\rho} \cdot \tilde{\rho} \right]^{n} \left[ (\tilde{H} - \tilde{\sigma}_j) \cdot \tilde{\rho} \right]^{m}
\]

(1.1)

Other aberration-based methods to deal with rotationally non-symmetric systems have also been developed. A similar vector-based method for plane-symmetric systems was developed by Jose Sasian [12,21]. Sasian’s method expands the wavefront aberrations about the OAR and employs symmetry constraints using a symmetry vector to simplify the expansion. Stone and Forbes developed methods employing Hamiltonian optics to describe the first-order properties of non-symmetric optical systems, including image plane tilt [22–24]. This method was later employed to a global optimization search of two, three, and four spherical mirror designs by Howard and Stone, where aberrations were optimized using RMS spot size [11,25,26]. Howard and Stone employed a similar method to a three-mirror freeform design, again using RMS spot-size optimization to control aberrations above first-order [27]. The main difference between these two paradigms and the paradigm of NAT is that in NAT, there are no “new” aberrations in the sense that the pupil dependence of the aberrations does not change, but instead each aberration acquires new field dependence.
We employ the methods of NAT in this work primarily because of the relatively easier conceptual understanding it provides. In Chapter 2, we use NAT to analyze and design a four-mirror laser relay. In Chapter 3, we expand NAT beyond sixth order to facilitate freeform design.

Fuerschbach et al. expanded NAT to include systems with FRINGE Zernike freeform surfaces to create the Aberration Theory of Freeform Surfaces (ATFS) [17]. ATFS facilitates the use of freeform surfaces in the optical design of rotationally non-symmetric optical systems. ATFS predicts the NAT aberrations of a given optical surface shape described by Zernike polynomial terms based on the surface’s location relative to a local pupil.

ATFS has been used recently in the design of many optical systems ranging from freeform spectrometers to head-worn displays to freeform telescopes [3,5,28,29]. A general design method employing visual analysis of full-field displays of the aberrations to guide the design process by Bauer et al. provides a framework for integrating freeform surfaces into the design process of imaging systems [30]. In Chapter 3, we add a qualitative aspect to the analysis of the full-field displays based on NAT and ATFS.

1.3 Two unobscured optical design regimes

In optical design, there is often a trade-off between complexity and performance. Typically, stricter system requirements such as large FOVs, wider apertures, or strict packaging constraints require more complexity. In this work, we consider two different regimes of unobscured reflective optical design requiring different yet related design
considerations due to their different applications and requirements, which ultimately yield
different requirements for the complexity of the surface shapes. The two design regimes
are characterized by their FOV and aperture requirements as shown in Table 1.1. The first
design is an all-reflective unobscured laser relay with a small FOV. This design type is
used at the University of Rochester’s Laboratory for Laser Energetics for the Achromatic
Image Relay (AIR), which will be discussed in Chapter 2.

Due to the small FOV of the AIR, it is natural to assume the surfaces can be simple
conics. If used appropriately, reflective conic sections can be used to image a single field
point stigmatically, without aberration. A system of such conic sections can image a single
point perfectly, but is limited in FOV. Such a system could be made unobscured by
offsetting the aperture, as shown in Section 2.3. Additionally, we show that the effective
radius of curvature of the off-axis sections of these conics has a simple relationship to the
Gaussian curvature, allowing us to use simple first-order imaging equations to hold certain
layout requirements for the AIR.

High-power laser-grade off-axis conic sections, however, can be costly to
manufacture, and present complicated alignment challenges. Therefore, one might consider
simplifying the design even further by using only spherical surfaces. By tilting the surfaces,
the design can be made unobscured. As we show in Section 2.4, such a design can be
analyzed with NAT
Table 1.1: Select specifications for the WFOV freeform imager and Achromatic Image Relay for comparison

<table>
<thead>
<tr>
<th></th>
<th>Achromatic Image Relay</th>
<th>WFOV Three-mirror Imager</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entrance Pupil Diameter</td>
<td>45 x 45 mm (square)</td>
<td>250 mm</td>
</tr>
<tr>
<td>Full Field-of-View (diagonal)</td>
<td>0.14 degrees</td>
<td>3.75 degrees</td>
</tr>
<tr>
<td>Focal Length</td>
<td>A focal</td>
<td>750 mm</td>
</tr>
<tr>
<td>Volume</td>
<td>3.0x1x0.05m–150 L</td>
<td>50-110 L</td>
</tr>
</tbody>
</table>

In contrast to the AIR, we consider the recently popular three-mirror form of a wide field-of-view telescope, which has been a popular design form for improvement using freeform surfaces [31–34]. It is well known that allowing the optical surface shapes to break from rotational symmetry adds degrees of freedom to a given design. These additional degrees of freedom can be used to tailor various aspects of the design, including improved imaging performance and a reduction in system volume. However, it would be useful to understand quantitatively what exactly the performance benefit is, be it optical performance or system volume, for a given design. To answer this question, we focused on a three-mirror form called the Three Mirror Compact (TMC), also known as the reflective triplet, which is regarded as the most compact three-mirror telescope form. In Chapter 4, we compare TMC designs using traditional off-axis sections of rotationally symmetric parent surfaces and designs using freeform surfaces. Specifically, we determine how much optical performance is improved for a given volume target, and how much the volume can be reduced for a given performance target for the TMC design.

1.4 Dissertation Outline

In Chapter 2, we present two works concerning the AIR. We first outline the requirements of the AIR. We then discuss two different designs for the AIR. The first is the dual-Cassegrain design, which is a rotationally symmetric design with an offset aperture that utilizes off-axis sections of conic mirrors. We then discuss the design and test-
bed alignment of a tilted component telescope design, which utilizes tilted spherical surfaces.

In Chapter 3, we discuss a method of analysis for designing freeform telescope designs using ATFS. We show how to determine the field-dependence of Zernike astigmatism up to 8\textsuperscript{th} order in NAT. We then show how to use this field-dependence to implement the reduction of certain aberrations using Zernike surface shapes based on ATFS.

In Chapter 4, we investigate the effect of freeform surfaces on the volume and performance of a type of three-mirror telescope, the three-mirror compact (TMC), also known as the reflective triplet. We compare designs using off-axis sections of rotationally symmetric surfaces and designs using freeform surfaces, and how these surface types affect the achievable volume for a given performance target and the performance at a given volume target.
2 Unobscured four-mirror laser relay

In this chapter, we discuss two different types of unobscured reflective relay designs: a dual-Cassegrain design using a series of confocal conic sections, and an all-spherical design using tilted spherical components. Both design types require the same first-order properties, discussed first in Section 2.2. The confocal conic sections have certain properties that allow us to use the “normal” first-order equations even in an offset aperture off-axis configuration, which we discuss in Section 2.3. The design theory and test-bed alignment of the tilted-component telescope (TCT) design type is discussed in Section 2.4.

Note that this chapter is a composition of two different manuscripts, one for each design type listed above. As such, the motivation for each design type is similar and shares some introductory material, motivations for the designs, and the first-order calculations.

2.1 Introduction to the AIR

The Laboratory for Laser Energetics (LLE) at the University of Rochester is developing an ultra-broadband optical parametric chirped-pulse amplifier laser system [35]. This laser, called the Multi-Terawatt Optical Parametric Amplifier Line (MTW-OPAL) [1], is designed to produce 7.5 J, 15 fs pulses with a 180 nm bandwidth from 830 nm to 1010 nm through multiple stages of noncollinear optical parametric amplifiers (NOPAs) [36]. Image relays between NOPAs preserve the beam quality and sequentially magnify the beam size according to its amplified energy level to keep the maximum fluence below the damage threshold. The final relay after the last NOPA stage (referred to as NOPA5 as it is the fifth amplifier) requires an all-reflective, unobscured
optical relay to avoid introducing additional longitudinal chromatic aberrations, also known as radial group delay (RGD) [37–39]. RGD refers to the fact that refractive image relays delay the pulse in the center of the beam with respect to the pulse in the outer edge of the beam as the center beam goes through thicker material in the lens. This relative group delay is on the order of hundreds of femtoseconds so it is significantly larger than the ideal design pulse width of 15 fs and would greatly reduce the focused intensity. The final all-reflective image relay, also referred to as an achromatic image relay (AIR), removes the longitudinal chromatic aberrations and RGD by avoiding lenses in favor of using mirrors. The AIR for NOPA5 performs two roles: it acts as a beam expander to create a 90 x 90 mm beam from the 45 x 45 mm output from NOPA5, and it relays the NOPA5 (N5) output to the fourth grating (G4) in the grating compressor chamber (GCC). Both requirements help prevent damage on G4 by lowering laser intensity and providing a uniform beam profile. These two roles must be achieved simultaneously, and thus there are two sets of conjugate planes that must be simultaneously realized: the “far-field” (FF) collimation planes and the “near-field” (NF) pupil imaging planes. This simultaneous dual-conjugate imaging case has been investigated for two refractive elements and four refractive elements by Wang et al. [40].

The required FOV for the far-field of the AIR is quite small, since the purpose of the AIR is to transport a beam with inherently low spatial frequency content, so it is natural to consider the confocal conic Cassegrain configuration: a parabola as the primary and hyperbola as the secondary. Two telescopes with this configuration placed back-to-back result in an aberration-free afocal beam expander for a single field point. To avoid
obscuration in this type of system, an offset aperture is required, resulting in the use of off-axis sections of each parent conic surface. If the parent mirrors share a common optical axis, then off-setting the FF aperture will maintain stigmatic imaging in the FF. This type of design is considered in Section 2.3.

In optical design, there is often a trade-off between complexity and performance. Off-axis conic sections are the most obvious choice for unobscured, small-FOV designs like the AIR, as we have previously reported [41]. However, it is important to understand if we can achieve our goals with simpler spherical optics since they reduce complexity and therefore cost. Instead of field-bias or aperture offset as in an off-axis conic design, one can tilt the optical components to avoid obscuration. Buchroeder details many examples of tilted component telescopes (TCT), some of which have only spherical components [10,42]. These designs use tilted components to avoid obscuration, but the relative tilts must be chosen to balance the resulting aberrations. Steven and Dubra describe the design of a two-mirror relay using tilted spherical components [16]. A four-mirror TCT design is considered in Section 2.4.

2.2 First-order optics

Whether designing a TCT or a system of confocal conics, the first-order optical properties are the same. There are nine first-order parameters for this 4-mirror system: the input and output working distances, \( d_0 \) and \( d_4 \), the three distances between the mirrors, \( d_1, d_2, \) and \( d_3 \), and the four radii of curvature of each mirror, \( c_1, c_2, c_3, c_4 \). The magnification constraint, NF imaging requirement, and FF collimation requirement determine three of these variables. The other six variables are available to meet the layout and space
requirements. To meet these requirements, the five distances are specified within a certain range based on the space constraints in the GCC, leaving only one truly free variable.

2.2.1 First-order 4-mirror solution

To solve for the constraints, we set up a system of equations using the paraxial optical matrix method [43]. The transfer matrix \( T_i \), describing the ray transfer between surfaces \( d_i \) and \( d_{i+1} \), with index of refraction \( n_i \) between surface \( i \) and \( i+1 \) is

\[
T_i = \begin{pmatrix} 1 & d_i / n_i \\ 0 & 1 \end{pmatrix}.
\]  \hspace{1cm} (2.1)

The reflection matrix \( R_i \) at a surface \( i \) with index difference \( \Delta n_{i,i+1} = n_i - n_{i+1} \) (\( \Delta n_{i,i+1} = \pm 2 \) for a reflective surface) and surface curvature \( c_i \) is

\[
R_i = \begin{pmatrix} 1 & 0 \\ c_i \Delta n_{i,i+1} & 1 \end{pmatrix}.
\]  \hspace{1cm} (2.2)

Note that the sign convention for these matrices is such that the index changes sign after reflection, and surfaces with positive curvature have the center of curvature to the right of the surface vertex. Additionally, positive distances are measured from left to right along the propagation axis.

The total system matrix from the input plane at N5 to the output plane at G4 can be built up through matrix multiplication of each transfer matrix:

\[
M_{N5,G4} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = T_4 R_4 T_3 R_3 T_2 R_2 T_1 R_1 T_0
\]  \hspace{1cm} (2.3)
The transfer matrix $T_2$ can be defined as the product of two transfer matrices $T_{2a}$ and $T_{2b}$, where $T_{2a}$ spans from just after mirror M2 to the intermediate focus and $T_{2b}$ spans from the intermediate focus to just before M3:

$$T_2 = T_{2b} T_{2a}.$$  (2.4)

The total system matrix can be built up by multiplying the optical matrices of its sub-systems, namely the optical matrix of the first two mirrors, M1 and M2, and the optical matrix of the second two mirrors, M3 and M4. The first sub-system matrix is

$$M_{1,2} = T_{2b} R_2 T_1 T_0 = \begin{bmatrix} 1 + 2c \beta + 2d \beta, & d_a + g_c & 2d_\alpha d_\beta, & c_a + 2g_c d_\alpha \\ 2c_\alpha & g_1 + 2d_\alpha c_\alpha \\ \end{bmatrix}. $$ (2.5)

Three quantities have been used in Eq. (2.5) to shorten the expressions and are defined as:

$$c_a \equiv c_1 - c_2 - 2c_1 c_2 d_1,$$  (2.6)

$$d_a \equiv d_0 + d_1 + 2c_1 d_0 d_1,$$  (2.7)

$$g_1 = (1 - 2c_2 d_1).$$  (2.8)

The second sub-system matrix is

$$M_{3,4} = T_4 R_4 T_3 T_{2b} = \begin{bmatrix} 1 + 2c \beta + 2d \beta, & d_\beta + g_2 & c_\beta + 2d_\alpha d_\beta, & c_\beta \\ 2c_\beta & g_2 + 2d_\alpha c_\beta \\ \end{bmatrix}. $$ (2.9)

Again, three quantities have been used in Eq. (2.9) for brevity:

$$c_\beta \equiv c_3 - c_4 - 2c_3 c_4 d_3,$$  (2.10)

$$d_\beta \equiv d_3 + d_2 + 2c_3 d_3 d_2.$$  (2.11)
\( g_2 = (1 - 2c_4 d_4) \)  

(2.12)

To ensure that the system is afocal, or equivalently, that an infinitely far away plane in object space is conjugate with the infinitely far plane in image space, we constrain \( C = 0 \) in Eq. (2.3). This is the FF imaging constraint. To ensure that the N5 plane is conjugate with the G4 plane, we constrain \( B = 0 \) in Eq. (2.3). This is the NF imaging constraint. Finally, to ensure that the afocal magnification is equal to \( m \), we constrain \( A = m \) (note that \( m \) is negative for an afocal system with a single internal image). Since an optical matrix is unitary, this is equivalent to the constraint \( D = 1 / m \) given the other two constraints. Thus, to apply the constraints, the system matrix is

\[
M_{N5,G4} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} m & 0 \\ 0 & 1 / m \end{bmatrix}.
\]

(2.13)

The distances \( d_0, d_1, d_2, d_3, d_4 \) are specified based on the layout constraints of the AIR in the GCC. The remaining first-order variables are the four curvatures \( c_1, c_2, c_3, c_4 \). Applying the NF, FF, and afocal magnification constraints results in the following equations for the curvatures of M2, M3, and M4.

\[
c_2 = \frac{m^2 [d_0 + d_2 + 2 c_4 d_3] + m d_4 + 2 c_1 (d_4 + d_3) d_4}{2 d_2 [m^2 d_0 + d_4 + 2 c_1 d_3 d_4]}
\]

(2.14)

\[
c_3 = \frac{m^2 d_0 + m (d_2 + d_4) + d_3 + 2 c_1 d_3 d_4}{2 d_4 d_3 m}
\]

(2.15)

\[
c_4 = \left[ 1 + \frac{d_3 + 2 c_1 d_3 d_4 + m d_2}{d_4 + 2 c_1 d_3 d_4 + m^2 d_0} \right] \frac{1}{2 d_4}
\]

(2.16)
Once the five distances $d_0, d_1, d_2, d_3, d_4$ and the curvature of $M_1, c_1$, are specified, the curvatures $c_2, c_3, c_4$ can be determined using Eqs. (2.14), (2.15), and (2.16), respectively.

### 2.3 Dual-Cassegrain and Gaussian Curvature

In this section, we derive the relationship between Coddington's equations and the Gaussian curvature (GC) for a stigmatic reflective imaging system. This relationship allows parameterizing off-axis conic optical systems using traditional first-order optics by considering the effective curvature at the center of the off-axis sections.

Specifically, we demonstrate parameterizing the system requirements of a 2x achromatic image relay. This system required both collimation (far-field) and pupil imaging (near-field) simultaneously. Long working distances and specific spatial constraints limited the available layout options for the imaging components. By parameterizing these system requirements and packaging constraints, the final specifications could be quickly iterated while allowing for flexibility in the layout of the system during a multi-year conceptual period. This work was published in Ref [41].

Unobscured mirror systems require breaking symmetry. Most optical designs employ an offset in aperture, a bias in the field-of-view, or both to an otherwise rotationally symmetric design to avoid obscuration [44]. The parent optical system of these designs is composed of the parent optical surfaces oriented along a common optical axis. In optical design software, the parent optical surfaces are specified using their axial parameters (e.g., vertex radius of curvature, conic constant). This is a convenient way to specify the design due to the rotational symmetry of the parent system, but the unobscured subset of the parent...
design does not share rotational symmetry about the effective apertures of each optical surface. Similarly, the distances between the effective apertures of unobscured designs are different than the distances between the parent surfaces' vertices, especially for a system of confocal conics. Packaging constraints are specified relative to the used apertures, not the parent apertures. In contrast to the co-axial rotationally symmetric designs in the present work, other recent design techniques for freeform telescopes employ a system of non-coaxial off-axis confocal conics combined with XY polynomials to correct field aberrations, breaking the axial symmetry of the design [21]. Here, we parameterize the properties of the coaxial parent surfaces in terms of the off-axis parameters to enable the rapid iteration of the design according to changing packaging requirements and specifications.

Due to the extremely small field-of-view requirements for the AIR (0.14 degrees), each telescope can use a Cassegrain configuration, which consists of a positive parabolic reflector and a negative hyperbolic reflector [45,46]. These conic surfaces of revolution are Cartesian reflectors when imaging at their stigmatic conjugates. A Cartesian reflector is a reflecting surface that perfectly images a point to a point. A Cassegrain is a system of coaxial confocal conics, which is a system of Cartesian reflectors whose stigmatic foci overlap.
To avoid obscuration, an offset in aperture is used. For a system of confocal conics, any ray passing through the stigmatic foci can be considered a paraxial ray. Therefore, paraxial first-order imaging equations can be used to compute the imaging properties along that ray. Consequently, we can choose a ray that passes through the center of the offset aperture and determine the exact distances between the centers of the used apertures throughout the system. These distances are the "off-axis parameters" that we must specify for layout purposes of the MTW-OPAL relay.

To compute the axial parameters, we must determine the relationship between the specified off-axis parameters and the axial parameters (Figure 2.1). The effective off-axis curvature is a function of both the varying curvature of the conic surface and the angle of incidence of the off-axis chief ray. We show that this effective curvature is exactly the square root of the Gaussian curvature of the conic surface. We prove this property by using the toroidal form of Coddington's equations.
Next, in sections 2.3.1 and 2.3.2, we determine the relationships that will allow us to use paraxial equations (e.g. the paraxial matrix method [43]) with the required off-axis distances. These relationships allow us to translate the off-axis parameters to the parents' axial parameters. Specifically, we take the off-axis distances, plug them into the normal paraxial equations to determine the (off-axis) focal lengths, and finally translate those (off-axis) focal lengths to the axial focal lengths and distances. Thus, the off-axis distances and off-axis "effective" radii determine the axial distances and axial radii required of the parent system given some offset in aperture. This translation is possible because of the use of Cartesian reflectors. In section 2.3.3, we apply these relationships to the design of the AIR.

2.3.1 Coddington's Equations and Gaussian curvature

Coddington's equations describe the conjugate distances for the tangential and sagittal rays at a given oblique angle of incidence [47,48]. They are given as

\[
\frac{n'}{s'} - \frac{n}{s} = \frac{n'\cos I' - n\cos I}{R_s},
\]

\[
\frac{n'\cos^2 I'}{t'} - \frac{n\cos^2 I}{t} = \frac{n'\cos I' - n\cos I}{R_t},
\]

where \( s \) and \( t \) are the sagittal and tangential object distances; \( s' \) and \( t' \) are the sagittal and tangential image distances; \( n \) and \( n' \) are the indices in the object and image spaces respectively; \( I \) and \( I' \) are the angle of incidence and refraction; and \( R_s \) and \( R_t \) are the sagittal and tangential radii of curvature at the point of intersection of the chief ray and the surface.
In the case of a mirror, we make the substitutions $n' = -1, n = 1, I' = -I$, and $s = t = \ell$. In the case of stigmatic imaging, $s' = t' = \ell'$, where we define $\ell, \ell'$ as the object and image conjugate distances (see Figure 2.2). Equations (2.17) and (2.18) then simplify as

$$\frac{1}{\ell'} + \frac{1}{\ell} = \frac{2\cos I}{R_j}, \quad (2.19)$$

$$\frac{1}{\ell'} + \frac{1}{\ell} = \frac{2}{R_j \cos I}. \quad (2.20)$$

With these simplifications, the relationship between $R_j$ and $R_i$ for a given angle of incidence $I$ is given as [49]:

$$\frac{R_j}{R_i} = \cos^2 I. \quad (2.21)$$

This is the relationship between the sagittal and tangential radii of curvature for astigmatism-free imaging for a given angle of incidence. Cartesian reflectors (conic reflectors operating at their stigmatic imaging conjugates) are surfaces of revolution, which means that these radii of curvature are in fact the principal radii of curvature (the reciprocals of the principal curvatures) because the plane of reflection lies on a meridional plane [50].

Equations (2.19) and (2.20) resemble the familiar Gaussian imaging equation for a mirror. Since $\ell$ and $\ell'$ are known from the first-order properties, we can consider an effective radius of curvature (RoC), $R_{\text{eff}}$, for the stigmatic off-axis conjugates given as

$$\frac{1}{\ell'} + \frac{1}{\ell} = \frac{2\cos I}{R_{\text{eff}}} = \frac{2}{R_j \cos I}. \quad (2.22)$$
From (2.22), we can write the effective RoC for a given angle of incidence of the chief ray on the Cartesian surface as

$$R_{\text{eff}} = \frac{R_s}{\cos I} = R_i \cos I.$$  \hspace{1cm} (2.23)

This effective RoC is the RoC to be used in the first-order imaging equations with the desired off-axis distances as the paraxial distances.

The Gaussian curvature is the product of the principal curvatures, which can be found from $\kappa_i = 1/R_i$ and $\kappa_2 = 1/R_s$. Applying the relationship in Equation (2.21), the Gaussian curvature for a Cartesian reflector is given as

$$K_G = \kappa_1 \kappa_2 = \frac{1}{R_s R_i} = \frac{\cos^2(I)}{R_s^2} = \frac{1}{R_s^2 \cos^2(I)}.$$  \hspace{1cm} (2.24)

Comparing Eqs. (2.23) and (2.24), we see that the effective RoC is related to the Gaussian curvature by

$$R_{\text{eff}} = \pm \frac{1}{\sqrt{K_G}}.$$  \hspace{1cm} (2.25)

Note that, except for planar surfaces, $K_G > 0$ for Cartesian reflectors. With this relation, the simple first-order imaging equations can be used to compute the imaging properties along any stigmatic ray in a stigmatic optical system made up of Cartesian reflectors.
2.3.2 Finding the coaxial curvature of a hyperboloid from the off-axis parameters

The required relay for MTW-OPAL can be achieved with two Cassegrain telescopes placed back to back. Figure 2.1 shows this configuration in an example layout with a chief ray traced through the system. The distances along the axis of symmetry (between the vertices $V_1, V_2$, etc.) are used to specify the system in the usual paraxial imaging equations (and in ray-tracing software). However, we now have a framework to use those same equations with the off-axis distances we wish to specify in our design.

For a function $z = F(x, y)$, the Gaussian curvature is given as

$$K_G = \frac{F_{xx} \cdot F_{yy} - F_{xy}^2}{(1 + F_x^2 + F_y^2)^2}, \quad (2.26)$$

where $F_i = \frac{\partial F}{\partial i}$ and $F_{ij} = \frac{\partial^2 F}{(\partial i \partial j)}$.

The sag of a conic surface of revolution is given by

$$z(r) = a \left[ \sqrt{1 + \frac{r^2}{aR_c}} - 1 \right], \quad (2.27)$$
where \( a \) is the distance from the geometric center to the surface vertex (see Figure 2.2), \( R_v \) is the vertex RoC, \( z \) is the axial coordinate or sag, and \( r \) is the radial (cylindrical) coordinate. This is not the typical sag equation for conics in optics, but the geometric parameter \( a \) lends itself better to computation. The conic constant is related to \( a \) by
\[
k = -(R_v/a + 1).
\]

Using Equation (2.26) with Equation (2.27), the Gaussian curvature \( K_o \) of a conic surface as a function of cylindrical coordinate \( r \) relates to the vertex RoC \( R_v \) and center-to-vertex distance \( a \) as
\[
K_o = \left[ \frac{aR_v}{r^2R_v + a(r^2 + R_v^2)} \right]^2
\tag{2.28}
\]

Equation (2.28) combined with equation (2.25) gives the relationship between the off-axis "effective curvature" we require and the associated axial curvature for a conic defined by \( a \).

In this work we are concerned with \( K_o \) for a hyperboloid, but Equation (2.28) can be generalized to any conic of revolution. For an ellipsoid, \( a \) is the semi-major axis measured from the geometric center to the vertex. Note that for an ellipsoid, \( a \) and \( R_v \) are always opposite in sign, and for a hyperboloid they are the same sign. For a paraboloid, \( a \to \infty \), so the limit of Equation (2.28) and Equation (2.27) must be taken.
For a hyperboloid, $a$ can be calculated from the known quantities $\ell$ and $\ell'$, which are the off-axis conjugate distances as shown in Figure 2.2. In fact, that is the definition of a hyperboloid: it is the set of points such that $||\ell'|-|\ell||=2|a|$.

Solving equation (2.28) and substituting $K_G$ according to equation (2.25) for the effective RoC $R_{\text{eff}}$, the vertex radius of curvature for a conic with an effective RoC $R_{\text{eff}}$ at radial coordinate $r$ is given as

$$R_v(r; a, R_{\text{eff}}) = \frac{aR_{\text{eff}} - r^2 + \sqrt{(aR_{\text{eff}} - r^2)^2 - 4a^2r^2}}{2a} \tag{2.29}$$

By expressing $a$ and $R_{\text{eff}}$ in terms of the angle of incidence $I$, the angle $\eta$ (Fig. 2.2), and $r$, equation (2.29) reduces to a more compact expression for $R_{v,h}$:

$$R_v = r \frac{\cos I}{\sin \eta} \tag{2.30}$$

Referring to Figure 2.2, $R_v$ can be determined by extending the local surface normal to the axis of symmetry of the surface of rotation. The center of curvature of the sphere tangent to the surface at $A$ in the sagittal direction is located at the intersection of the surface normal and the axis of symmetry. We can then compute the RoC of this sphere to obtain $R_v$ as

$$R_v = \frac{r}{\sin(\eta)} = \frac{r}{\sin\left(\frac{\alpha - \beta}{2}\right)} \tag{2.31}$$

With this result, equation (2.30) can be further simplified to
\[ R_x = R_y \cos I = R_{\text{eff}} \cos^2 I \]  \hspace{1cm} (2.32)

Note that this result is independent of any parameter regarding the specific shape of the conic surface and therefore applies to any Cartesian reflector.

### 2.3.3 Application to the design process

The relationship between the Gaussian curvature of a Cartesian surface and the stigmatic imaging condition along with first-order parameterization allows us to directly design the relay mirror system for the MTW-OPAL upgrade. The layout constraints inside the grating compressor chamber (GCC) in which this relay resides are typically in flux and require constant shifting of the optical layout to conform to the new requirements. In addition, MTW-OPAL is a research laser system designed to support engineering technologies applicable to the larger beam size of such as used in the OMEGA Laser System at LLE, therefore requirements would need to be scaled.
The relay is required to take collimated light from the N5 plane to the G4 plane with 2x magnification (see Figure 2.3) while imaging the N5 plane onto the G4 plane.

To parameterize the first-order constraints of the NF and FF imaging conditions simultaneously, we set up a system of equations using a 2x2 rotationally-symmetric matrix following the first-order optical matrix method and then impose the required imaging conditions on the system matrix [43]. The matrices span from the N5 plane to the G4 plane and contain nine variables: the five off-axis distances $d_0, d_1, d_2, d_3, d_4$ and the off-axis effective focal lengths (EFL’s) of the mirrors $f_1, f_2, f_3, f_4$. Applying the NF, FF, and afocal magnification constraints results in the following equations for the curvatures of M2, M3, and M4. From the curvatures, $f_2, f_3,$ and $f_4$ can be calculated as $f_i = 1/2c_i$.

The five distances are specified to conform to the spatial constraints inside the GCC (see Figure 2.3). The EFL of the first mirror $f_1$ is chosen to maintain a certain angle of
incidence given the aperture offset. The other three focal lengths are determined by the NF imaging, FF imaging, and magnification conditions (see Equations (2.14), (2.15), and (2.16)). The EFL's determine the effective RoC $R_{\text{eff}}$ for each mirror for use in equation (2.32), which determines the vertex radius $R_v$ for each mirror.

The shape parameter $a$ is calculated from $\|\ell'|-|\ell||=2|a|$ for the hyperboloids. The shape parameter $a$ is therefore not a degree of freedom as it is imposed by the combined distances $\ell$ and $\ell'$. With $a$ in hand, we use equation (2.29) to obtain $R_v$. For the paraboloids, $a \to \infty$, so use the limit of equation (2.29) as $a \to \infty$. Once the vertex radii $R_v$ and conic shape parameters are determined for all mirrors, the geometry is entirely specified due to the imposed stigmatic imaging condition. The locations of the stigmatic foci relative to the vertices can be calculated from $R_v$ and $a$, and then the axial distances between the vertices can be calculated from the foci locations.

A suite of programs was written in MATLAB to quickly iterate the design using the above calculations. Two examples with different layout requirements (one long and one short) are shown in Figure 2.4 and Figure 2.5. The off-axis requirements and solved axial parameters for both designs are given in Tables 2.1 and 2.2, respectively. The FF entrance pupil diameter is 45 mm and the FF full FOV is 0.14 degrees.
Table 2.1. Required off-axis parameters for the long and short image relays used to compute the axial parameters given in Table 2.2.

<table>
<thead>
<tr>
<th></th>
<th>Long</th>
<th>Short</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>d0</td>
<td>2250</td>
<td>800</td>
<td>mm</td>
</tr>
<tr>
<td>d1</td>
<td>130</td>
<td>80</td>
<td>mm</td>
</tr>
<tr>
<td>d2</td>
<td>2800</td>
<td>500</td>
<td>mm</td>
</tr>
<tr>
<td>d3</td>
<td>500</td>
<td>300</td>
<td>mm</td>
</tr>
<tr>
<td>d4</td>
<td>5500</td>
<td>1000</td>
<td>mm</td>
</tr>
<tr>
<td>f1</td>
<td>592.772</td>
<td>174.345</td>
<td>mm</td>
</tr>
<tr>
<td>f2</td>
<td>-713.688</td>
<td>-184.727</td>
<td>mm</td>
</tr>
<tr>
<td>f3</td>
<td>-534.349</td>
<td>-873.575</td>
<td>mm</td>
</tr>
<tr>
<td>f4</td>
<td>892.863</td>
<td>527.263</td>
<td>mm</td>
</tr>
</tbody>
</table>

Table 2.2. Computed axial parameters corresponding to the required off-axis parameters in Table 2.1. These parameters describe the designs in Figs. 2.4 and 2.5.

<table>
<thead>
<tr>
<th>CodeV Command</th>
<th>Long System</th>
<th>Short System</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>YDE</td>
<td>340</td>
<td>100</td>
<td>mm</td>
<td>Aperture offset in +y direction</td>
</tr>
<tr>
<td>THI S'm1'</td>
<td>2303.601</td>
<td>515.765</td>
<td>mm</td>
<td>N5-M1 distance</td>
</tr>
<tr>
<td>THI S'm2'</td>
<td>-131.765</td>
<td>-76.641</td>
<td>mm</td>
<td>M1-M2 distance</td>
</tr>
<tr>
<td>THI S'focus'</td>
<td>1399.351</td>
<td>292.511</td>
<td>mm</td>
<td>Intermediate focus to M3 distance</td>
</tr>
<tr>
<td>THI S'm3'</td>
<td>-427.254</td>
<td>-294.962</td>
<td>mm</td>
<td>M3-M4 distance</td>
</tr>
<tr>
<td>THI S'm4'</td>
<td>5657.12</td>
<td>1019.702</td>
<td>mm</td>
<td>M4-G4 distance</td>
</tr>
<tr>
<td>RDY S'm1'</td>
<td>-1078.34</td>
<td>-317.159</td>
<td>mm</td>
<td>Radius of Curvature, M1</td>
</tr>
<tr>
<td>RDY S'm2'</td>
<td>-1203.75</td>
<td>-300.231</td>
<td>mm</td>
<td>Radius of Curvature, M2</td>
</tr>
<tr>
<td>RDY S'm3'</td>
<td>791.456</td>
<td>1556.39</td>
<td>mm</td>
<td>Radius of Curvature, M3</td>
</tr>
<tr>
<td>RDY S'm4'</td>
<td>1471.486</td>
<td>1015.121</td>
<td>mm</td>
<td>Radius of Curvature, M4</td>
</tr>
<tr>
<td>K S'm1'</td>
<td>-1</td>
<td>-1</td>
<td></td>
<td>Conic constant, M1</td>
</tr>
<tr>
<td>K S'm2'</td>
<td>-3.82072</td>
<td>-7.09728</td>
<td></td>
<td>Conic constant, M2</td>
</tr>
<tr>
<td>K S'm3'</td>
<td>-2.45106</td>
<td>-39.9524</td>
<td></td>
<td>Conic constant, M3</td>
</tr>
<tr>
<td>K S'm4'</td>
<td>-1</td>
<td>-1</td>
<td></td>
<td>Conic constant, M4</td>
</tr>
</tbody>
</table>

Both example relays have Gaussian image planes at the required NF and FF locations simultaneously through the imposed constraints on the first-order equations. Stigmatic imaging for the far-field is ensured through the use of the conic surfaces with coincident stigmatic foci. The distances along the central chief ray match the specified distances in Table 2.1 while avoiding obscuration using an offset aperture. The RMS wavefront performance of these examples was computed in CODE V®. The RMS WFE is less than 0.000068 waves and 0.000041 waves for the short and long relays, respectively, at a wavelength of 910 nm.
2.3.4 Conclusion for Section 2.3

We have shown the connection between Gaussian curvature and stigmatic imaging. This connection was revealed using Coddington's equations and the definition of the Gaussian curvature. This relationship was then exploited to directly design an unobscured image relay for a high-power laser using Cartesian reflectors. To directly express the axial parameters of the relay in terms of the off-axis constraints, we used the relationship between Gaussian curvature and the effective radius of curvature of the off-axis imaging system. We included an example of a long and a short relay with different distance constraints to illustrate the flexibility of the algorithm.

2.4 Unobscured tilted-component telescope for AIR

To achieve the AIR requirements with an all-spherical tilted component design, the same first-order parameterization of Section 2.2 is applied. However, we no longer have the advantage of the conic shape to correct for aberrations. Instead, the tilt angles of the mirrors can be adjusted to mitigate the off-axis aberrations that result from tilting the components.

The all-spherical design starts with a first-order design that conforms to the layout and imaging requirements. The tilts of the mirrors are adjusted to simultaneously avoid obscuration and provide adequate imaging performance for both the near- and far-field cases at the appropriate magnification.
We show the theoretical basis for the correction of field-constant coma and field-constant astigmatism using two of the mirror tilts. We then show the alignment of the design in a test-bed setup.

2.4.1 Space constraints

The space constraints inside the GCC dictate the allowed ranges of the five distances for the AIR design (discussed in Section 2.2.1). Figure 2.6(a) shows the contents of the GCC, including the AIR and the grating compressor. Figure 2.6(b) shows the layout of the final design of the AIR alone on the GCC optical table without the other components for clarity. The locations of the AIR mirrors are limited by the other contents and beam paths within the GCC.

Table 2.3 shows the values of $d_0$ to $d_4$ allowed by the GCC constraints, the input value of the M1 curvature, and the resulting curvatures of M2, M3, and M4 ($c_2, c_3, c_4$) from Eqs. (2.14)-(2.16). The input and output working distances, $d_0$ and $d_4$ respectively, are dictated by the relative positions of the input plane (N5), the space designated for the AIR inside the GCC, and the location of the output plane (G4). The optical pulse compressor after AIR is made up of four gratings and is an unfolded design of the original Treacy compressor [51]. G4 is the fourth grating.
Figure 2.6. (a) The contents of the GCC. The AIR mirrors are labelled M1, M2, M3, and M4 and the AIR is outlined and shaded in red. The grating compressor is outlined in blue. The final grating of the grating compressor is labelled G4. The input plane for the AIR is the NOPA5 crystal, labelled N5. The chief ray of the beam path is shown in red. (b) The GCC with everything except the AIR removed, with the beam path still shown. (c) The test-bed setup to test the alignment of the AIR. The pinhole jig locations are labeled as Pn. The chief ray of the beam path is shown in red.
Table 2.3. The input and output values of the paraxial equations for the AIR. The values $d_0, d_1, d_2, d_3, d_4, c_1, c_2, c_3, c_4$ are the inputs and $c_2, c_3, c_4$ are the output values.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Paraxial Value (mm)</th>
<th>Final Design Value (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_0$</td>
<td>2261.7176</td>
<td></td>
</tr>
<tr>
<td>$d_1$</td>
<td>-250.0000</td>
<td></td>
</tr>
<tr>
<td>$d_2$</td>
<td>3200.0000</td>
<td></td>
</tr>
<tr>
<td>$d_3$</td>
<td>-717.1000</td>
<td></td>
</tr>
<tr>
<td>$d_4$</td>
<td>5194.9600</td>
<td></td>
</tr>
<tr>
<td>$1/c_1$</td>
<td>-3180.0000</td>
<td></td>
</tr>
<tr>
<td>$1/c_2$</td>
<td>-19377.622</td>
<td>-19146.7</td>
</tr>
<tr>
<td>$1/c_3$</td>
<td>1775.9790</td>
<td>1759.5</td>
</tr>
<tr>
<td>$1/c_4$</td>
<td>2587.5567</td>
<td>2580.0</td>
</tr>
</tbody>
</table>

The damage threshold of the coatings on the mirrors translates to a minimum beam size of 35 x 35 mm on M2 and M3, which creates a “stay-out zone” for these two mirrors. This places a constraint on the curvature of M1, $c_1$, for a given M1-to-M2 separation distance, $d_1$, as shown in Figure 2.7. Paraxially, this constraint is given as

$$c_1 > \frac{D_1 - D_2}{2d_1D_1}.$$  \hspace{2cm} \text{(2.33)}

$D_1$ is the beam diameter on M1 (45 mm) and $D_2$ is the beam diameter on M2. For the given value of $d_1$, this gives a minimum concave RoC for M1 of 2250 mm. However, the curvature of M1 must also be tuned such that the solution to the other curvatures also keeps M3 out of the stay-out zone.

Figure 2.7. Illustration of the minimum beam size requirement on M2 and M3 given by Eq. (2.33). $F_1$ is the focal length of M1. D1 and D2 are the beam diameters on M1 and M2, respectively. $d_1$ is the signed distance between M1 and M2.
2.4.2 Unobscured tilted component telescopes

With the first-order parameters determined, the next step was to create an unobscured configuration. To do so, we can tilt the components appropriately. However, tilting spherical components results in aberrations. Buchroeder showed that these types of designs, known as tilted-component telescopes, can correct the aberrations due to the tilts by appropriately tuning the tilts of the mirrors [19,52]. In addition, Rogers showed the same using nodal aberration theory (NAT) for two and three mirrors for extended fields-of-view [53]. Because the field-of-view for the AIR is small at 0.14 degrees, we are able to achieve the required correction while maintaining enough freedom in the design to keep the AIR within the space constraints of the GCC.

Given the minimum tilt angle of M1 required to remove the obscuration of M2, we can use the theoretical framework of NAT to fully cancel the field-constant astigmatism ($B_{222}$), field-linear astigmatism ($A_{222}$), and field-constant coma ($A_{131}$) using the three tilts of M2, M3, and M4 [53]. However, this solution produces designs that are not fully unobscured and do not fit into the GCC space. Fortunately, because of the small field and slow F-number, linear astigmatism is not a significant aberration for the AIR. Therefore, we can use the tilts of M3 and M4 to cancel the $A_{131}$ and $B_{222}$ caused by unobscuring M1 and M2.

Using the framework of NAT, we can predict the M3 and M4 tilts necessary to correct the field-constant astigmatism and field-constant coma induced from the tilts of M1 and M2. The system of equations involves two NAT aberration terms composed in terms of the
sigma vectors of each surface. The sigma vectors are then related to the tilts of each surface \( i \).

\[
\hat{A}_{131} = \sum_{i=1}^{4} \hat{A}_{131,i} = \sum_{i=1}^{4} W_{131,i} \sigma_i 
\]

(2.34)

\[
\hat{B}_{222} = \sum_{i=1}^{4} \hat{B}_{222,i} = \sum_{i=1}^{4} W_{131,i} \sigma_i^2 
\]

(2.35)

From these two equations, we can solve for the values of \( \sigma_3 \) and \( \sigma_4 \) that simultaneously null both field-constant coma and field-constant astigmatism. We can see from the sigma dependence that the solutions will be the solutions to a quadratic equation, and thus there will be two potential configurations. The solutions for \( \sigma_3 \), \( \sigma_4 \) are given below.

\[
\sigma_3 = - \frac{\hat{A}_{131,\{1,2\}} W_{222,4} W_{131,3} \pm W_{131,4} \sqrt{-F}}{W_{222,4} W_{131,3}^2 + W_{222,3} W_{131,4}^2} 
\]

(2.36)

\[
\sigma_4 = \frac{-\hat{A}_{131,\{1,2\}} W_{222,3} W_{131,4} \pm W_{131,3} \text{sign}(W_{131,4}) \sqrt{-F}}{W_{222,4} W_{131,3}^2 + W_{222,3} W_{131,4}^2} 
\]

(2.37)

\[
F \equiv A_{131,\{1,2\}} W_{222,3} W_{222,4} + B_{222,\{1,2\}}^2 \left( W_{222,4} W_{131,3}^2 + W_{222,3} W_{131,4}^2 \right) 
\]

(2.38)

In the above equations, the subscript \( \{1,2\} \) denotes the sum as defined in Eqs. (2.34) and (2.35) for the first two surfaces, M1 and M2. For the first-order values listed in Table 2.3 and the given angles of M1 and M2, the output angles of the solution in Eqs. (2.36) and (2.37) are given in Table 2.4. Both angle solutions are given, but only one solution is unobscured.
Table 2.4. Angles of AIR mirrors for three designs: the two solutions to Eqs. (2.36) and (2.37)

<table>
<thead>
<tr>
<th>Angle (degrees)</th>
<th>Solution 1</th>
<th>Solution 2</th>
<th>Final Design</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>-8.2815</td>
<td>15.276</td>
<td></td>
</tr>
<tr>
<td>M2</td>
<td>9.0740</td>
<td>-6.8394</td>
<td>9.100</td>
</tr>
<tr>
<td>M3</td>
<td>-3.9475</td>
<td>2.2906</td>
<td>-3.959</td>
</tr>
</tbody>
</table>

Figure 2.8 and Figure 2.10 show the layouts of the solution 1 and solution 2 designs, respectively. Figure 2.9 and Figure 2.11 show the full-field displays of the wavefront error for solutions 1 and 2, respectively. We can see that both solutions produce a design free of field-constant coma and field-constant astigmatism. However, solution 2 remains partially obscured due to M3, and its maximum RMS WFE is more than twice as large, therefore we proceed with solution 1. However, there is a better balance of all the aberrations if we depart slightly from the exact solution. The M3 and M4 angles of solution 1 along with the radii of M2, M3, and M4 were further optimized to produce a more optimal wavefront. The final design with numerically optimized radii and angles is also given in Table 2.3 and Table 2.4, respectively.
2.4.3 AIR Alignment Test-bed

An alignment plan for the AIR has been developed and tested. The alignment procedure consists of two stages: course alignment and fine alignment. The course alignment stage uses a precision coordinate measuring machine (CMM) arm to place precision-machined pinhole jigs. The CMM arm is a FARO Gage Plus, from which we achieved on the order of 100 micron placement accuracy of the pinholes [54]. Passing a pencil beam from an alignment laser diode through the pinholes establishes a line which serves as the optical axis. Since the mirrors are spherical, guiding the pencil beam through the pinholes using tip/tilt and z-travel on each mirror can be used to position the center-of-curvature close to the nominally designed position. All four mirrors can be coarsely aligned sequentially with this method. Tolerance analysis assuming 100-micron accuracy in the mirror placement of the AIR design shows that the coarse alignment using the pinholes is adequate to position the mirrors accurately enough such that fine alignment of M4 tip, tilt,
and defocus can correct any residual defocus and astigmatism still present after course alignment. Additionally, M3 tip/tilt can be used to correct residual coma, if necessary.

To test the feasibility of the alignment plan and to troubleshoot possible difficulties before the final installation in the GCC, we developed a test-bed to replicate the in-situ alignment before the final system is aligned at LLE. The essential components of the test-bed are a 532 nm laser diode, a custom wave-front sensor [55], a polarizing beam splitter, a beam expander, and several planar mirrors.

The NOPA5 crystal outputs a 45 x 45 mm collimated beam. To replicate this configuration in the test-bed, we use two beam expanders as shown in Figure 2.12. The first beam expander is a 4x beam expander that focuses the output of a laser diode into a 25-micron pinhole used as a spatial filter. The resulting Gaussian beam is then collimated and truncated using a square apodizer before passing through a polarizing beamsplitter (PBS). The second beam expander then enlarges the beam to the 45 x 45 mm beam size. There is a quarter wave plate at the focus of the 10x beam expander to rotate the polarization of the return beam from vertical to horizontal in order to reduce spurious reflections from the beam splitter. Additionally, an adjustable iris is used to create a pencil beam for the course alignment step. The return beam reflects off the PBS interface to a wavefront sensor and a blank mirror substrate with a wedge acting as a second beam-splitter. The second return beam is directed into a focusing lens and CCD camera used for maintaining alignment.
Figure 2.12. The test-bed components to produce a 45 x 45 mm beam and to measure the return wavefront. WFS – Wavefront sensor. Cam – Camera. QWP – quarter wave plate.

### 2.4.4 Course Alignment

The first step in the course alignment process is to align the input beam to the pinholes, which are shown in Figure 2.6(c). The pinhole is a hole in a metal plate that rests kinematically inside the pinhole jig. The pinhole jig has a post-hole for a camera to aid with alignment of the beam through the pinhole. The input beam was guided through the first pinhole, P1, with the tip/tilt of the first planar fold mirror. A second fold mirror before P1 guided the beam towards the second pinhole, P2. This process was iterated until the beam passed through both pinholes. A CCD array sensor was used to aid in determining the uniformity of the beam so that the beam passed through the center of the pinholes accurately.

The process to align the spherical mirrors is similar. The locations of the pinholes and mirrors in the test-bed is shown in Figure 2.6(c). These locations were reproduced from the locations in the GCC to approximate the in-situ alignment process. To align the first spherical mirror, M1, the mirror mount was first placed using the CMM in its nominal position. Then a three-actuator mirror mount was used to tip, tilt, and translate along the beam to guide the beam through P3 and P4, the third and fourth pinholes. Since a spherical
mirror’s position is described by the position of its center-of-curvature (CoC), these three
degrees of freedom are sufficient to position the CoC such that it passes the beam through
both pinholes. A tip/tilt of a spherical mirror is equivalent to translating the CoC. This
process was repeated for M2, M3, and finally M4.

After this course alignment, the full beam was passed through the system, auto-
reflected from a planar mirror, and passed back through the system to the wavefront sensor
to achieve a double-pass setup as seen in Figure 2.6c. The course alignment alone was not
sufficient to produce a well-collimated beam, as expected.

Figure 2.13. Adjustments for M3 and M4. Red arrows indicate actuator adjustments. Blue arrows
indicate the motion of the mirror.

2.4.5 Fine Alignment

The first step in the fine alignment process was to collimate the output of the AIR. The
M4 mirror mount was itself mounted on a linear translation stage as shown in Figure 2.13.
Using feedback from the wavefront sensor, M4 was translated along the incoming beam
using the translation stage to adjust the output collimation to achieve a large RoC wavefront (~100 m) at the wavefront sensor plane.

After this initial collimation, M4 was finely aligned using the three mirror mount actuators to adjust tip, tilt, and z-translation as shown in Figure 2.13. M4 tip and tilt were used to minimize Zernike astigmatism (Z5/6). Fine z-translation was adjusted by rotating all three actuators by the same amount and was used to minimize Zernike power (Z4).

M4 fine alignment required an iterative approach to minimize power and astigmatism. To speed up the alignment process, the general alignment algorithm seen in Figure 2.14 was used. To start, the wavefront was fit with FRINGE Zernike terms up to Z37 using a circular aperture circumscribing the square beam. To predict the required adjustment for a given actuator, the actuator was rotated by some known amount and the wavefront was measured again to determine the change in the target Zernike mode. Assuming a linear relationship between the target Zernike mode and the actuator motion, we predicted the required actuator change to minimize the Zernike mode (e.g. drive it to zero). Once the given Zernike mode was minimized, we performed the same process for the next Zernike mode.

This general approach was repeated for each actuator/Zernike term pair. For M4, the X-tilt actuator was used to minimize the Z5 term (0° astigmatism), the Y-tilt actuator was used to minimize the Z6 term (45° astigmatism), and all three actuators turned by the same amount were used to minimize the Z4 mode (Zernike power).
Once these terms were sufficiently minimized, the residual aberrations were inspected. The remaining dominant aberrations were Z7/Z8 (coma), Z9 (spherical), and higher order terms. Minimization of Z7/Z8 terms requires two more degrees of freedom. Tip and tilt on M3 were used to adjust Z7/Z8 using the same iterative process as astigmatism. However, because an adjustment of tip/tilt on M3 also induces astigmatism (Z5/6), we must readjust M4 before examining the change in Z7/Z8 values. Thus, an actuator on M3 was rotated, then the M4 astigmatism minimization process was repeated, and then Z7/8 magnitude change is recorded to predict the required M3 actuator change. This process is repeated until Z7/8, Z5/6, and Z4 are simultaneously driven to sufficiently small values to produce a satisfactory wavefront.

![Algorithm for adjusting a single actuator to minimize a given Zernike mode.](image)

To account for the double-pass setup, the wavefront values are divided by a factor of two. Additionally, to account for the difference between the test-bed wavelength of 532 nm and the nominal central wavelength of the laser operating bandwidth of 910 nm, we
divide the wavefront values by 1.71, bringing the total reduction factor to 3.42. The performance target of the in-situ AIR is less than 0.07 waves RMS and 0.25 waves PV at a wavelength of 910 nm. Thus, the target wavefront error in the test-bed was required to be less than 0.24 waves of RMS WFE and less than 0.89 waves PV at the test wavelength of 532 nm.

2.4.6 Alignment results

To aid in the alignment process, a MATLAB program was constructed to record the output of the wavefront sensor in the test-bed at each fine alignment step, record the changes made to the alignment actuators at each step, and linearly extrapolate the changes to predict the amount of a given actuator adjustment required to reduce a given Zernike term to zero. In practice, a single actuator adjustment produced changes to more than one Zernike mode, even for small changes, for a number of reasons. First, even in an idealized setup, tilting spherical mirrors induces linear astigmatism, field-constant astigmatism, and field-constant coma, as well as higher order aberrations. Second, the actuators do not produce idealized tip/tilt/defocus motion relative to the center of curvature of the mirrors, but instead produce some combination of all three degrees of freedom. Furthermore, the wavefronts are averaged over many samples due to the turbulence of the air in the test-bed. Additionally, the beam is a square profile as the beam in MTW-OPAL is square to maximize amplifier efficiency, whereas Zernike modes are orthogonal only over a circular aperture. Therefore, we can expect some level of degeneracy in the fit, and we expect changes in multiple Zernike modes for a given actuator adjustment. Nonetheless, the
designation of each actuator adjustment to each Zernike term produced acceptable results when used iteratively.

![Figure 2.15](image_url) The double pass wavefront given in waves at 532 nm for (a) after the course alignment step and (b) after the fine alignment steps. Note that the color scale ranges from \([0,2]\) waves for (a) and from \([0,1]\) waves for (b).

The subsequent wavefront after the course alignment routine using the pinholes is shown in Figure 2.15(a). The dominant aberration in the wavefront after the course alignment step appears to be defocus, as shown by the Zernike mode decomposition of the wavefront shown in Figure 2.16. A description of each alignment step is shown in Table 2.5. Figure 2.17 and Figure 2.18 show the Peak-to-Valley (PV) and root mean square (RMS) wavefront error (WFE) of the wavefront at each step in the fine alignment process. The wavefront in Figure 2.15a corresponds to step 1 in Figure 2.16 and in Table 2.5. Referring to Figure 2.16, we see that the coefficient for Zernike defocus (Z4) is 0.67 waves. This defocus was more than predicted by the tolerances of the CMM coarse alignment stage, a clue to another issue that will be addressed in the next section.

Figure 2.15(b) shows the measured wavefront error after fine alignment. There is residual spherical aberration, as predicted by the raytracing model. Additionally, there is still some coma (Z7/8) present and higher-order modes are also becoming dominant. However, since we have reached the desired PV and RMS WFE targets, we stopped the
alignment process. With the aid of the program used to decompose the wavefront and predict the adjustments seen in Table 2.5, the fine alignment routine can be completed in less than 1 hour. After completing fine alignment, the resulting wavefront PV at 532 nm was 0.61 waves, and the RMS WFE was 0.10 waves in double-pass. Applying the reduction factor of 3.42 to the measured wavefronts in Figure 2.15, we have 0.176 waves PV and 0.029 waves RMS. Thus, we can expect a wavefront PV of less than 0.2 waves and RMS WFE less than 0.05 waves at 910 nm at the output of the AIR using this method.

Figure 2.16. The FRINGE Zernike mode coefficient values at each step for terms Z4 to Z9 given in waves at 532 nm.
Figure 2.17. The peak-to-valley and root-mean-square (rms) values of the wavefront at each step.

Figure 2.18. The rms wavefront error values of the wavefront at each step at a smaller scale to show detail.
Table 2.5. Description of each fine alignment step, including the amount that the given actuator was adjusted and the amount that is predicted for that actuator

<table>
<thead>
<tr>
<th>Step</th>
<th>Adjustment Type</th>
<th>Actuator Adjustment Amount (degrees)</th>
<th>Predicted Adjustment Amount (degrees)</th>
<th>Target Zernike Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>M4 all</td>
<td>100.0</td>
<td>523.8</td>
<td>Z4 (defocus)</td>
</tr>
<tr>
<td>2</td>
<td>M4 all</td>
<td>480.0</td>
<td>-106.7</td>
<td>Z4 (defocus)</td>
</tr>
<tr>
<td>3</td>
<td>M4 all</td>
<td>-100.0</td>
<td>-6.6</td>
<td>Z4 (defocus)</td>
</tr>
<tr>
<td>4</td>
<td>M4 x tilt</td>
<td>90.0</td>
<td>-31.9</td>
<td>Z5 (astigmatism)</td>
</tr>
<tr>
<td>5</td>
<td>M4 x tilt</td>
<td>-30.0</td>
<td>-425.6</td>
<td>Z5 (astigmatism)</td>
</tr>
<tr>
<td>6</td>
<td>M4 all</td>
<td>-60.0</td>
<td>20.7</td>
<td>Z4 (defocus)</td>
</tr>
<tr>
<td>7</td>
<td>M3 y tilt</td>
<td>-216.0</td>
<td>-2700</td>
<td>Z8 (coma)</td>
</tr>
<tr>
<td>8</td>
<td>M4 y tilt</td>
<td>-90.0</td>
<td>-58.4</td>
<td>Z6 (astigmatism)</td>
</tr>
<tr>
<td>9</td>
<td>M4 y tilt</td>
<td>-50.0</td>
<td>-300.0</td>
<td>Z6 (astigmatism)</td>
</tr>
<tr>
<td>10</td>
<td>M4 y tilt</td>
<td>-80.0</td>
<td>28.0</td>
<td>Z6 (astigmatism)</td>
</tr>
<tr>
<td>11</td>
<td>M4 y tilt</td>
<td>0.0</td>
<td>0.0</td>
<td>Z6 (astigmatism)</td>
</tr>
<tr>
<td>12</td>
<td>M4 all</td>
<td>40.0</td>
<td>135.5</td>
<td>Z4 (defocus)</td>
</tr>
<tr>
<td>13</td>
<td>M4 all</td>
<td>130.0</td>
<td>-58.7</td>
<td>Z4 (defocus)</td>
</tr>
<tr>
<td>14</td>
<td>M4 all</td>
<td>-60.0</td>
<td>12.3</td>
<td>Z4 (defocus)</td>
</tr>
<tr>
<td>15</td>
<td>M3 y tilt</td>
<td>360.0</td>
<td>-5101.4</td>
<td>Z8 (coma)</td>
</tr>
<tr>
<td>16</td>
<td>M4 y tilt</td>
<td>90.0</td>
<td>378.0</td>
<td>Z6 (astigmatism)</td>
</tr>
<tr>
<td>17</td>
<td>M4 y tilt</td>
<td>360.0</td>
<td>23.2</td>
<td>Z6 (astigmatism)</td>
</tr>
<tr>
<td>18</td>
<td>M4 all</td>
<td>-90.0</td>
<td>25.7</td>
<td>Z4 (defocus)</td>
</tr>
<tr>
<td>19</td>
<td>M3 y tilt</td>
<td>360.0</td>
<td>1023.5</td>
<td>Z8 (coma)</td>
</tr>
<tr>
<td>20</td>
<td>M4 y tilt</td>
<td>360.0</td>
<td>121.2</td>
<td>Z6 (astigmatism)</td>
</tr>
<tr>
<td>21</td>
<td>M4 y tilt</td>
<td>120.0</td>
<td>-20.2</td>
<td>Z6 (astigmatism)</td>
</tr>
<tr>
<td>22</td>
<td>M4 x tilt</td>
<td>60.0</td>
<td>-8.4</td>
<td>Z5 (astigmatism)</td>
</tr>
<tr>
<td>23</td>
<td>M4 all</td>
<td>-90.0</td>
<td>-33.1</td>
<td>Z4 (defocus)</td>
</tr>
<tr>
<td>24</td>
<td>M4 all</td>
<td>-30.0</td>
<td>-56.8</td>
<td>Z4 (defocus)</td>
</tr>
<tr>
<td>25</td>
<td>M3 x tilt</td>
<td>-180.0</td>
<td>-482.4</td>
<td>Z7 (coma)</td>
</tr>
<tr>
<td>26</td>
<td>M4 x tilt</td>
<td>120.0</td>
<td>65.3</td>
<td>Z5 (astigmatism)</td>
</tr>
<tr>
<td>27</td>
<td>M4 x tilt</td>
<td>0.0</td>
<td>-0.0</td>
<td>Z5 (astigmatism)</td>
</tr>
<tr>
<td>28</td>
<td>M4 all</td>
<td>-90.0</td>
<td>-316.7</td>
<td>Z4 (defocus)</td>
</tr>
<tr>
<td>29</td>
<td>M4 all</td>
<td>-300.0</td>
<td>14.3</td>
<td>Z4 (defocus)</td>
</tr>
<tr>
<td>30</td>
<td>M4 x tilt</td>
<td>60.0</td>
<td>-5.2</td>
<td>Z5 (astigmatism)</td>
</tr>
<tr>
<td>31</td>
<td>M4 x tilt</td>
<td>-40.0</td>
<td>7.2</td>
<td>Z5 (astigmatism)</td>
</tr>
<tr>
<td>32</td>
<td>Final</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
</tbody>
</table>

2.4.7 Magnification measurement

After the wavefront was aligned, the magnification of the beam was measured. Minimizing magnification error is important for laser systems with large beams. For the AIR, the magnification was required to be -2.00 ± 2%. To measure the magnification in the test-bed, a mask with holes was placed at the object plane and the distance between the images of the holes was measured in the image. The relative distance between the holes was measured.
The result was a magnification of \(-2.13 \pm 2\%\), which is a \(+7.5\%\) change from the required value of \(-2.00 \pm 2\%\). At first, this error was suspected to be due to uncertainties in the alignment process. However, the course and fine alignment processes were repeated multiple times to test this hypothesis, and the magnification was re-measured each time. The measured value of the magnification varied by less than 1\% between alignments.

Combined with the defocus error after course alignment seen in Section 2.4.6, the magnification error suggested errors in the radii of curvature of the AIR test-bed mirrors. The radii of the two concave mirrors, M1 and M4, were measured using an optical cavity setup as described in [56] and found to be \(3152 \pm 2\) mm and \(2549 \pm 6\) mm, respectively. The radii of the convex parts were more difficult to measure accurately, especially the almost 20-m convex radius of M2. A similar differential cavity measurement technique was applied, but instead using an F/7.2 focusing reference sphere, which has a concave surface to provide positive power to the cavity. The resulting radii measurements were \(19600 \pm 600\) mm and \(1760 \pm 30\) mm for M2 and M3, respectively. Using these radii in the nominal optical model and re-optimizing the position of M4 and the tip/tilt of M3 to simulate our alignment, we see an increase in magnification of \(~5\%\). Combined with the alignment uncertainty from the course alignment step, this explains most of the error in magnification seen in the test-bed. In retrospect, the magnification specification tolerance was checked only against alignment errors, and not radius of curvature errors. The radii were given a tolerance of \(\pm 1\%\), which was sufficient to allow correction of the wavefront in alignment. However, the ensure that the magnification would be within spec, the
tolerance analysis would need to be re-run to determine the tightened tolerances on the radii.

![Figure 2.19. Direction of motion of M3 (indicated by the red arrow) to compensate magnification error.](image)

To bring the magnification back into spec, the position of another mirror besides M4 must be adjusted. The same space constraints in the design process also restrict us here, however. M2 is already very close to the GCC wall as seen in Figure 2.6a. M1 has some room to move, but moving M1 would require changing the input angle of the beam from N5, which is not ideal. M3, being the only mirror left, is the best candidate to move. To test how far was required, the location of M3 was moved by 75 mm towards M2 to shorten the focal length of the M3-M4 pair in the model with the measured radii, which reduced the magnification by 3.5%. This adjustment reduced the output working distance to the image plane (G4) by 75 mm. This change was reproduced in the test-bed and the magnification was re-measured to be 2.07, bringing the magnification error down from 7.5% to 3.5%. Based on the model, to bring the magnification error down another 3.5%, we could move the M3 mirror by another 75 mm. Since the test-bed optics likely do not have exactly the same radii as the in-situ AIR optics, this final M3 mirror adjustment was not performed on the test-bed. The actual compensation will be based on the measurements of the in-situ AIR mirrors.
2.4.8 Conclusion to Section 2.4

The design of an unobscured reflective laser relay comprising four tilted spherical mirrors has been described. We showed the theoretical basis for this four-mirror design configuration using first-order optical matrix methods and Nodal Aberration Theory. We then described the process of aligning the design in a test-bed to demonstrate an effective alignment method for this geometry type. We were able to achieve a nominal design that met specifications. We were also able to successfully align the design in a test-bed environment to achieve our target wavefront error of less than 0.25 waves PV and 0.07 waves RMS. We discussed a current challenge identified through the development of the testbed to simultaneously meet a magnification requirement, which can be remediated in the in-situ AIR optics.
3 NAT-based ratio method of freeform design

When using freeform surfaces in optical design, the field dependence of the aberrations can become quite complex, and understanding these aberrations facilitates the design process. Here, we calculate the field dependence of low-order Zernike astigmatism (Z5/6) up to 8th order in Nodal Aberration Theory (NAT). Expansion of NAT astigmatism terms to 8th order facilitates a more accurate fit to the Zernike astigmatism data. We then show how this estimated field dependence can be used to quantitatively analyze a freeform telescope design. This analysis tool adds to the optical designer’s arsenal when up against the challenge of designing with freeform optics.

3.1 Introduction

The rising popularity of freeform optics has driven the creation of new design methods and aberration theories that can account for a lack of rotational symmetry. It is often convenient to represent the aberrations of a given optical system in terms of Nodal Aberration Theory (NAT). To facilitate the use of freeform surfaces in optical design of rotationally non-symmetric optical systems, Fuerschbach et al. developed an extension to NAT, here called the Aberration Theory of Freeform Surfaces (ATFS) [17]. The ATFS predicts the aberrations of a given optical surface shape described by Zernike polynomial terms based on its location relative to a local pupil. The ATFS, methods derived from it, and other non-rotationally symmetric aberration descriptions make use of Full-Field Displays (FFDs). FFDs show the magnitude and orientation of a given aberration at discrete points in the field-of-view (FOV) [21,30,53]. Bauer et al. show how to guide the
design process using a visual, designer guided semi-quantitative assessment by examining Zernike FFDs to ascertain the aberration content of the optical system [30]. This method allows the designer to compare off-axis folding geometries and to decide which Zernike surface terms should be varied at a given step in the design based on the FFDs using the ATFS and NAT. In this work, we present a supporting method of freeform design based on both the principles in the ATFS and on the methods of Bauer et al. that uses a quantitative numerical analysis of the NAT aberration content from the underlying FFD data. This method is aided by an expansion of NAT to 8th order for a more robust fit to the higher-order field dependent aberrations that contribute to the Zernike astigmatism (Z5/6) FFDs.

To that end, we first expand the NAT wavefront expansion to 8th order for terms that contribute higher-order field dependence to the Zernike astigmatism FFD. We then show that the field-dependence of Zernike astigmatism can be determined by decomposing the FFD data into its constituent NAT aberration terms. Finally, we show how this can be used to determine the value of the surface coefficients required to correct certain NAT aberrations using the ATFS.

3.2 Expansion of NAT astigmatism to 8th order

Nodal Aberration Theory (NAT) uses the vector formulation of the Hopkins wavefront expansion theorized by Shack and developed by Thompson with contributions from Buchroeder [18,19,57,58]. Individual NAT polynomial terms are determined by expanding the wavefront in terms of the field and pupil coordinate vectors and the sigma vector, which
describes the aberration field centers. The general NAT wavefront in vector notation is given by Eq. (3.1), adapted from Thompson \[59, Eq. 3.2 \].

\[
W = \sum_{j=0}^{N} \sum_{p=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [W_{kwm}] \left( \vec{H} - \vec{\sigma}_j \right) \left( \vec{H} - \vec{\sigma}_j \right)^T \left( \vec{\rho} - \vec{\sigma}_j \right) \left( \vec{\rho} - \vec{\sigma}_j \right)^T \left( \vec{H} - \vec{\sigma}_j \right) \left( \vec{\rho} - \vec{\sigma}_j \right)
\]

Thompson provides descriptions of spherical, coma, and astigmatism NAT aberrations up to 6\textsuperscript{th} order in wavefront, where 6\textsuperscript{th} order includes those terms for which \( k + \ell \leq 6 \) \[59–62\]. Expanding this expression allows us to determine the field dependence of the various aberrations according to NAT. In design methods like Bauer \textit{et al}., a qualitative analysis of the aberration full-field displays (FFDs) is used to understand the field dependence of the aberrations. To simplify the task of visually analyzing an FFD and to reduce the burden on the designer, we propose an analytical and quantitative way to determine this field dependence using the NAT aberration polynomial, which we describe in Section 3.3. However, because higher-order terms begin to dominate when the low-order aberration terms are corrected, 6\textsuperscript{th} order NAT terms may be insufficient to accurately describe the field dependence and, therefore, expansion of Eq. (3.1) up to \( k + \ell = 8 \) will be useful. Yet, we are only interested in the terms that produce new types of higher-order field-dependence. This is because fitting the FFD data to the NAT polynomial does not distinguish between terms with different \( \sigma \) or \( \rho^2 \) (as opposed to \( \vec{\rho}^2 \), the vector quantity) dependence, only field-dependence \( \vec{H} \).

Note that in this work, we refer to the “order” of aberrations by the wavefront order, in contrast to the optical terminology of ray aberrations that considers wavefront
expansions to 4th order as “3rd-order” and 6th order expansions as “5th-order”, etc. The latter terminology refers to the combined aperture and field dependence of the ray aberration coefficients, which we do not consider in this work, so we have persisted with the wavefront order.

3.2.1 Field dependence of Zernike Astigmatism (Z5, Z6) up to 8th order

The first step in determining the 8th order NAT field dependence of the Zernike aberrations is to expand the NAT wavefront to 8th order. In this work, we focus on low-order Zernike astigmatism (Z5 and Z6 in FRINGE ordering) since it is often the largest aberration of off-axis or tilted component systems. Additionally, according to the ATFS, all low-order Zernike freeform surface shapes contribute to the field dependence of Zernike astigmatism, so it is critical to have a quantitative understanding of it for freeform design.

To analytically understand the astigmatism terms with higher-order field dependence, we can examine the difference between the NAT astigmatism terms for expansion through 6th order and through 8th order. The 6th order terms are those terms where \( k + \ell = 6 \), and 8th order have \( k + \ell = 8 \), which we have denoted in the subscripts.

\[
W_{k+\ell=6,m=2} = \sum_{j=1}^{n} W_{242, j} \left[ (\bar{h} - \bar{\sigma}_j) \cdot (\bar{\eta} - \bar{\sigma}_j) \right] \hat{\rho} \cdot \hat{\rho} \left[ (\bar{h} - \bar{\sigma}_j) \cdot \hat{\rho} \right]^2 \\
+ \sum_{j=1}^{n} W_{422, j} \left[ (\bar{h} - \bar{\sigma}_j) \cdot (\bar{\eta} - \bar{\sigma}_j) \right] \hat{\rho} \cdot \hat{\rho} \left[ (\bar{h} - \bar{\sigma}_j) \cdot \hat{\rho} \right]^2, \tag{3.2}
\]

\[
W_{k+\ell=8,m=2} = \sum_{j=1}^{n} W_{242, j} \left[ (\bar{h} - \bar{\sigma}_j) \cdot (\bar{\eta} - \bar{\sigma}_j) \right] \hat{\rho} \cdot \hat{\rho} \left[ (\bar{h} - \bar{\sigma}_j) \cdot \hat{\rho} \right]^2 \\
+ \sum_{j=1}^{n} W_{422, j} \left[ (\bar{h} - \bar{\sigma}_j) \cdot (\bar{\eta} - \bar{\sigma}_j) \right] \hat{\rho} \cdot \hat{\rho} \left[ (\bar{h} - \bar{\sigma}_j) \cdot \hat{\rho} \right]^2 \\
+ \sum_{j=1}^{n} W_{622, j} \left[ (\bar{h} - \bar{\sigma}_j) \cdot (\bar{\eta} - \bar{\sigma}_j) \right] \hat{\rho} \cdot \hat{\rho} \left[ (\bar{h} - \bar{\sigma}_j) \cdot \hat{\rho} \right]^2. \tag{3.3}
\]
It is worth noting here that Thompson treats $W_{42}$ as an oblique spherical aberration [60]. We include it here because, as Thompson also notes, it is in fact “normal” 4th-order astigmatism but with a 4th-order pupil dependence, and therefore contributes to the Z5/6 field dependence. Additionally, Thompson considers field curvature terms and astigmatism terms in the same treatment [62]. Because we consider here only terms contributing to Z5/6 field dependence, we do not consider the field curvature terms in the expansion.

The 6th order astigmatism expansion terms are generated from Eq. (3.2) and the 8th order astigmatism terms are generated from Eq. (3.3). Comparing the first lines and the second lines of each equation, we see that they produce the same field-dependence in the 6th order and 8th order since they differ only by a $\rho^2$ factor. Furthermore, we can see that higher-order field dependence will be produced by the third line of Eq. (3.3). We can expand this line to determine the new field dependent terms:

$$
\sum_{j=1}^{n} W_{622,j} \left[ (\vec{H} - \vec{\sigma}_j) \cdot (\vec{H} - \vec{\sigma}_j) \right]^{\frac{1}{2}} \left[ \vec{\rho} \cdot \vec{\rho} \right]^{\frac{1}{2}} \left[ (\vec{H} - \vec{\sigma}_j) \cdot \vec{\rho} \right]^{\frac{1}{2}}
$$

$$= \sum_{j=1}^{n} W_{622,j} \left[ H^2 + \sigma_j^2 - 2(\vec{H} \cdot \vec{\sigma}_j) \right]^{\frac{1}{2}} \left[ \vec{H} \cdot \vec{\rho} - \vec{\sigma}_j \cdot \vec{\rho} \right]^{\frac{1}{2}}$$

$$= \sum_{j=1}^{n} W_{622,j} \left[ H^4 + \sigma_j^4 + 4(\vec{H} \cdot \vec{\sigma}_j)^2 + 2H^2\sigma_j^2 - 4H^2(\vec{H} \cdot \vec{\sigma}_j) - 4\sigma_j^2(\vec{H} \cdot \vec{\sigma}_j) \right]...
$$

$$\times \left[ (\vec{H} \cdot \vec{\rho})^2 + (\vec{\sigma}_j \cdot \vec{\rho})^2 - 2(\vec{H} \cdot \vec{\rho})(\vec{\sigma}_j \cdot \vec{\rho}) \right] \quad (3.4)$$

At this stage, it is helpful to introduce a few specific Shack Vector Product (SVP) identities to allow us to group terms according to field dependence. The SVP is a geometric algebra operator in a Clifford algebra that is similar to the multiplication of complex numbers [63]. However, SVP eliminates the need to convert real vectors into complex
numbers to multiply them, and it therefore allows us to conveniently separate the wavefront expansion according field and pupil dependence. We adopt the same notation for the SVP as in Fuerschbach et al. [17], where two vectors are multiplied by SVP when they are adjacent to each other: \( \vec{H}\vec{\sigma} \). Additionally, a squared or cubed vector (except as noted below) such as \( \vec{H}^2 \) indicates a \( \vec{H}\vec{H} \).

The SVP identities below allow us to transform Eq. (3.4):

\[
\left( \vec{H} \cdot \vec{\rho} \right)^2 = \frac{1}{2} H^2 \rho^2 + \frac{1}{2} \bar{H}^2 \cdot \bar{\rho}^2 \tag{3.5}
\]

\[
\left( \vec{\sigma}_j \cdot \vec{\rho} \right)^2 = \frac{1}{2} \sigma_j^2 \rho^2 + \frac{1}{2} \sigma_j^2 \cdot \sigma_j^2 \tag{3.6}
\]

\[
\left( \vec{H} \cdot \vec{\sigma}_j \right)^2 = \frac{1}{2} H^2 \sigma_j^2 + \frac{1}{2} \bar{H}^2 \cdot \bar{\sigma}_j^2 \tag{3.7}
\]

\[
2 \left( \vec{H} \cdot \vec{\rho} \right) \left( \vec{\sigma}_j \cdot \vec{\rho} \right) = \left( \vec{H} \cdot \vec{\sigma}_j \right) \rho^2 + \left( \vec{H} \cdot \vec{\sigma}_j \right) \cdot \bar{\rho}^2 \tag{3.8}
\]

To further simplify the algebra, we can examine the Zernike Z5/6 polynomial and compare with the \( \rho \) dependence in Eq. (3.4) in light of Eqs. (3.5)-(3.8):

\[
Z_5 \propto \rho^5 \cos 2\phi = \bar{\rho}^2 \cdot \hat{x}, \tag{3.9}
\]

\[
Z_6 \propto \rho^6 \sin 2\phi = \bar{\rho}^2 \cdot \hat{y}. \tag{3.10}
\]

Any terms in Eq. (3.4) that do not have \( \bar{\rho}^2 \) dependence will not contribute to the \( Z_{5/6} \) FFD and, therefore, we can ignore them for the purposes of this exercise.
\[ [W]_{622}^{Z_{5/6}} = \sum_{j=1}^{n} W_{622,j} \left[ H^4 + \sigma_j^4 + 4H \cdot \sigma_j^2 + 2\vec{H} \cdot \vec{\sigma}_j^2 - 4H \cdot \left( \vec{H} \cdot \vec{\sigma}_j \right) - 4\sigma_j^2 \left( \vec{H} \cdot \vec{\sigma}_j \right) \right] \times \left[ \frac{1}{2} \vec{H} \cdot \vec{\rho}^2 + \frac{1}{2} \vec{\sigma}_j \cdot \vec{\rho}^2 - \left( \vec{H} \cdot \vec{\sigma}_j \right) \cdot \vec{\rho}^2 \right] \] (3.11)

We denote the wavefront expansion \([W]_{622}^{Z_{5/6}}\) with the superscript \(Z_{5/6}\) to refer to those NAT wavefront terms that contribute to the Zernike astigmatism aberrations, \(Z5\) and \(Z6\). The bracketed term \([W]_{622}\) is used to denote a shorthand for the NAT wavefront expansion for the terms related to the Hopkins \(w_{622}\) term, similar to the shorthand used in [59]. To dissect and better understand Eq. (3.11), it is instructive to gather the terms according to their “order” of field dependence and then put them all together at the end. The notation used here to symbolize the aberration terms according to \(\vec{\sigma}\) dependence, originally developed by K. Thompson in [58], can be found concisely in [17]. Note that terms such as \(\vec{c}_{622}^3\) are should not be confused with the cube of \(\vec{c}_{622}\). The superscript is meant to denote that the given term depends on different operations of \(\vec{\sigma}\), as can be seen in the definitions of each term in the following sections.

### 3.2.1.1 6th order field-dependence

Only one term will contribute 6th order dependence according to Eq. (3.11):

\[ \[W]_{622,6^{th}\text{Order}}^{Z_{5/6}} = \frac{1}{2} \sum_{j=1}^{n} W_{622,j} H^4 \vec{H} \cdot \vec{\rho}^2 \] (3.12)

Note that this term has no \(\vec{\sigma}_j\) dependence. This is simply the “normal” rotationally symmetric 8th order astigmatism term expressed in the appropriate NAT vector format.
3.2.1.2 \textbf{5th order field-dependence}

There are two terms that contribute to the 5th order field dependence:

\[
W_{622,5^{th}\text{ Order}} = -\sum_{j=1}^{n} W_{622,j} H^2 \left[ H^2 \left( \vec{H} \cdot \vec{\sigma}_j \right) + 2 \left( \vec{H} \cdot \vec{\sigma}_j \right) \vec{H}^2 \right] \cdot \vec{\rho}^2 .
\]  

(3.13)

We then define the 8th order NAT coefficient $\overline{A}_{622}$

\[
\overline{A}_{622} = \sum_{j=1}^{n} W_{622,j} \vec{\sigma}_j .
\]  

(3.14)

This gives us our final 5th order representation:

\[
W_{622,5^{th}\text{ Order}} = -\sum_{j=1}^{n} H^2 \left[ H^2 \left( \vec{H} \overline{A}_{622,j} \right) + 2 \left( \vec{H} \cdot \overline{A}_{622,j} \right) \vec{H}^2 \right] \cdot \vec{\rho}^2 .
\]  

(3.15)

3.2.1.3 \textbf{4th order field-dependence}

At first glance, there appears to be four 4th order terms:

\[
W_{622,4^{th}\text{ Order}} = \sum_{j=1}^{n} W_{622,j} \left[ \frac{1}{2} H^4 \vec{\sigma}_j^2 + 2 H^2 \sigma_j^2 \vec{H}^2 + \left( \vec{H} \cdot \vec{\sigma}_j \right) \vec{H}^2 + 4 H^2 \left( \vec{H} \cdot \vec{\sigma}_j \right) \left( \vec{H} \cdot \vec{\sigma}_j \right) \right] \cdot \vec{\rho}^2 .
\]  

(3.16)

We define three sigma dependences:

\[
B_{622} = \sum_{j=1}^{n} W_{622,j} \sigma_j^2 ,
\]  

(3.17)

\[
\overline{B}_{622} \approx \sum_{j=1}^{n} W_{622,j} \vec{\sigma}_j^2 ,
\]  

(3.18)

\[
\overline{B}_{622} = \sum_{j=1}^{n} W_{622,j} \left( \vec{\sigma}_j \right)^2 .
\]  

(3.19)
Combining these definitions and the identity Eqs. (3.68) and (3.69) in the Appendix with Eq. (3.16) yields the 4th order terms:

\[
W_{622,4\text{th}\text{Order}}^{Z_{66}} = \sum_{j=1}^{n} \left[ 3H^4 \bar{B}_{622,j}^2 + 4B_{622,j} H^2 \bar{H}^2 + \frac{1}{2} \bar{H}^4 \bar{B}_{622,j}^2 \right] \rho^2 .
\]  

(3.20)

### 3.2.1.4 3rd order field-dependence

There appears to be four 3rd-order terms:

\[
W_{622,3\text{rd}\text{Order}}^{Z_{66}} = \sum_{j=1}^{n} W_{622,j} \left[ -4H^2 \sigma_j^2 \left( \bar{H} \sigma_j \right) - 2 \bar{H} \cdot \sigma_j \left( \bar{H} \sigma_j \right) \cdots \right. \\
\left. - 2H^2 \left( \bar{H} \cdot \sigma_j \right) \sigma_j^2 - 2 \sigma_j^2 \left( \bar{H} \cdot \sigma_j \right) \bar{H}^2 \right] \rho^2 .
\]  

(3.21)

However, we can apply the identities in Eqs. (3.70)-(3.74) to simplify to just three terms. We define the following sigma dependent aberration coefficients:

\[
\bar{C}_{622}^3 = \sum_{j=1}^{n} W_{622,j} \bar{\sigma}_j^3 ,
\]  

(3.22)

\[
\bar{C}_{622}^2 = \sum_{j=1}^{n} W_{622,j} \sigma_j^2 \bar{\sigma}_j ,
\]  

(3.23)

\[
\bar{C}_{622}^* = \sum_{j=1}^{n} W_{622,j} \sigma_j^2 \bar{\sigma}_j^* .
\]  

(3.24)

Collecting all the terms considering the identities and definitions, we get the 3rd-order field dependent terms:

\[
W_{622,3\text{rd}\text{Order}}^{Z_{66}} = \sum_{j=1}^{n} \left[ 6H^2 \bar{H} \bar{C}_{622,j}^2 + 2H^2 \bar{H}^2 \bar{C}_{622,j}^* \right. \\
\left. + 2 \bar{H} \bar{C}_{622,j}^3 \right] \rho^2 .
\]  

(3.25)
### 3.2.1.5 2\textsuperscript{nd} order field-dependence

The second order has four terms that can be simplified to three terms while separating the field-dependence from the sigma dependence.

\[
W_{Z_{622}, 2\text{nd Order}}^{Z_{622}} = \sum_{j=1}^{n} W_{622,j} \left[ \frac{1}{2} \sigma_j^4 \vec{H}^2 + 2 H^2 \sigma_j^2 \vec{\sigma}^2 + \left( \vec{H}^2 \cdot \vec{\sigma}^2 \right) \sigma_j^2 + 4 \sigma_j^2 \left( \vec{H} \cdot \vec{\sigma} \right) \vec{H} \vec{\sigma} \right] \cdot \vec{p}^2. \tag{3.26}
\]

The first two terms are already separated. The 3\textsuperscript{rd} and 4\textsuperscript{th} terms can be separated and combined with the others using Eqs. (3.75) and (3.76). There are three sigma-dependent terms to define:

\[
D_{622} = \sum_{j=1}^{n} W_{622,j} \sigma_j^4, \tag{3.27}
\]

\[
\vec{D}_{622}^3 = \sum_{j=1}^{n} W_{622,j} \sigma_j^2 \vec{\sigma}_j^2, \tag{3.28}
\]

\[
\vec{D}_{622}^4 = \sum_{j=1}^{n} W_{622,j} \vec{\sigma}_j^4. \tag{3.29}
\]

Combining the terms and substituting the definitions, we get the second-order field-dependent terms:

\[
W_{Z_{622}, 2\text{nd Order}}^{Z_{622}} = \sum_{j=1}^{n} \left[ 3D_{622,j} \vec{H}^2 + 4 H^2 \vec{D}_{622,j}^2 + \frac{1}{2} \vec{H}^2 \vec{D}_{622,j}^4 \right] \cdot \vec{p}^2. \tag{3.30}
\]

### 3.2.1.6 1\textsuperscript{st} order field-dependence

\[
W_{Z_{622}, 1\text{st Order}}^{Z_{622}} = -\sum_{j=1}^{n} W_{622,j} \left[ \sigma_j^4 \left( \vec{H} \vec{\sigma}_j \cdot \vec{p}^2 \right) + 2 \sigma_j^2 \left( \vec{H} \cdot \vec{\sigma}_j \right) \left( \vec{\sigma}_j^2 \cdot \vec{p}^2 \right) \right]. \tag{3.31}
\]
The second term can be separated into two terms based on field and sigma dependence and combined with the first term using Eq. (3.77). We now have two 5th-order sigma dependent terms to define:

\[ \tilde{E}_{622}^{\sigma} = \sum_{j=1}^{n} W_{622,j} \sigma_j^{4} \sigma_j, \]  
(3.32)

\[ \tilde{E}_{622}^{\rho} = \sum_{j=1}^{n} W_{622,j} \sigma_j^{2} \sigma_j^{3}. \]  
(3.33)

Combining the terms and applying the definitions, we get the first-order field-dependent terms:

\[ W_{622,3^\text{rd}\text{Order}}^{Z_{\infty}} = -\sum_{j=1}^{n} \left[ 2 \tilde{E}_{622}^{\sigma} \tilde{H} + E_{622}^{3} \tilde{H}^{T} \right] \cdot \tilde{\rho}^{2}. \]  
(3.34)

3.2.1.7 Field-constant dependence

\[ W_{622,\text{Const}}^{Z_{\infty}} = \frac{1}{2} \sum_{j=1}^{n} W_{622,j} \sigma_j^{4} \left( \sigma_j^{3} \cdot \tilde{\rho}^{2} \right). \]  
(3.35)

There is no field-dependence to separate in this case, but we do need to define a new sigma-dependent term:

\[ \tilde{F}_{622}^{\rho} = \sum_{j=1}^{n} W_{622,j} \sigma_j^{2}. \]  
(3.36)

Thus, the field-constant 8th order term is given by:

\[ W_{622,\text{Const}}^{Z_{\infty}} = \sum_{j=1}^{n} \frac{1}{2} \tilde{F}_{622,j}^{\rho} \cdot \tilde{\rho}^{2}. \]  
(3.37)
Some of the terms from Eqs. (3.12), (3.15), (3.20), (3.25), (3.30), (3.34), and (3.37) have higher-order field dependence compared to the 6th order terms, but some are simply new combinations of sigma and field dependence. The relevant polynomial for the purposes of this work is constructed by collecting all the terms up to 6th order and adding in the new field-dependence from the 8th order expansion.

\[
W_{k/r/c/b}^{2,5/6} = \frac{1}{2} \left[ W_{222} + W_{422} H^2 + W_{622} H^4 - 2 \overline{H} \left( \overline{A}_{422} + 2H^2 \overline{A}_{622} \right) \right] \overline{H}^2 ...

+ 3H^2 \overline{B}_{422} + 6H^4 \overline{B}_{622} + \overline{H} \overline{B}_{822}^* ...

- 2\left( \overline{A}_{422} + H^2 (\overline{A}_{622} + 6\overline{C}_{622}) + H^4 \overline{A}_{822} \right) \overline{H} ...

- \overline{C}_{422} \overline{H}^* - 4H^2 \overline{H}^* \overline{C}_{622}^* - 4H^4 \overline{C}_{622} + \overline{B}_{822}^* \overline{H}^* + \overline{B}_{822}^* \right] \overline{H}^2 ...
\]

Note that only the lowest-order field-dependent term is included in Eq. (3.38), e.g. only the field-constant 4th order NAT term is included and the field-constant 6th order NAT term is excluded. The new field dependent terms from the 8th order expansion are in bold text in Eq. (3.38).

3.3 Fitting the Z_{5/6} FFD with NAT field dependence

Recent design methods relying on the ATFS use the Zernike FFD as a guide for the designer to determine the limiting aberrations in a given design and therefore determine the shape of the surface to use to correct this aberration [17,30]. These rely on a qualitative analysis of the FFD and the designer’s knowledge of the ATFS. Many high performance reflective freeform systems have been designed and even fabricated using this method [3–5,64]. However, the designer may find it advantageous to be able to determine exactly how much of a given NAT aberration is present in a system. Furthermore, armed with this
information, a designer could predict the magnitude of a given Zernike surface shape to add to a given surface. In this section, we will show it is possible using the field-dependence of the Zernike astigmatism FFD shown in Eq. (3.38).

To estimate the NAT aberrations in an optical system, we can fit the Zernike FFD data with the NAT polynomial. This method is similar to Gray et al. which performed a similar analysis for the standard rotationally symmetric wavefront expansion [65]. Said another way, we can expand the NAT wavefront polynomial in terms of the Zernike polynomials. The coefficients of each Zernike polynomial term will be determined by the following integral (adapted from [65]):

$$z_n\left( \vec{H}; \vec{\sigma} \right) = \frac{1}{N_n} \int_0^{2\pi} \int_0^1 \frac{W(\vec{H}, \vec{\rho}; \vec{\sigma}) Z_n(\vec{\rho}) \rho d\rho d\phi}{d_{\vec{\sigma}}^n}$$

Equation (3.39)

Here, we have used the FRINGE ordering of the Zernike polynomials [66]. We are interested in the coefficients of the astigmatism terms ($Z_{5/6}$) of the Zernike polynomial:

$$Z_5(\vec{\rho}) \propto \rho^2 \cos 2\phi = \vec{\rho}^2 \cdot \hat{x}$$

$$Z_6(\vec{\rho}) \propto \rho^2 \sin 2\phi = \vec{\rho}^2 \cdot \hat{y}$$

Equation (3.40)

In light of Eq. (3.40), the only terms in the NAT wavefront polynomial that will be non-zero following the integral of Eq. (3.39) for $Z_5$ and $Z_6$ will be terms that depend on the $\hat{x}$ or $\hat{y}$ components of $\vec{\rho}^2$, respectively. Because of the dot product with $\vec{\rho}^2$ in Eq. (3.38), all of the $\hat{x}$ components of the left side of that dot product will be non-zero in the $Z_5$ integral and the $\hat{y}$ components will be non-zero in the $Z_6$ integral. Therefore, for the
purposes of our fitting algorithm, it remains to determine the components of the left side of Eq. (3.38).

### 3.3.1 Plane symmetric optical systems

For plane symmetric optical designs that are symmetric about the $y$-$z$ plane, this task is greatly simplified by the fact that the sigma vectors are only along the $\hat{y}$ axis, and therefore every NAT coefficient has no $\hat{x}$ component. However, the polynomial in Eq. (3.38) does contain various vector operations involving the field coordinate $\vec{H}$ that need to be calculated in order to determine the vector components of each term.

\begin{align}
\vec{H} &= H_x \hat{x} + H_y \hat{y}, \\
\vec{H}^* &= -H_x \hat{x} + H_y \hat{y}, \\
\vec{H}^2 &= 2H_x H_y \hat{x} + \left(-H_x^2 + H_y^2\right) \hat{y}, \\
\vec{H}^{*2} &= -2H_x H_y \hat{x} + \left(-H_x^2 + H_y^2\right) \hat{y}, \\
\vec{H}^3 &= H_x \left(3H_y^2 - H_x^2\right) \hat{x} + H_y \left(H_x^2 - 3H_y^2\right) \hat{y}, \\
\vec{H}^4 &= -4H_x H_y \left(H_y^2 - H_x^2\right) \hat{x} + \left(H_x^4 - 6H_x^2 H_y^2 + H_y^4\right) \hat{y}.
\end{align}

This is all the SVP mathematics that we need to know for a plane symmetric system. The rest are either inner products or scalar operations. The products between the field coordinates and the NAT coefficients are simplified by the fact that each coefficient simply acts as a scalar since it only has a $\hat{y}$ component.

There are two additional simplifications to the terms in Eq. (3.38) due to plane symmetry.
\[ B_{622}^2 = B_{622}^{*2} = |B_{622}^2| \hat{y} \]  
\[ C_{622}^3 = C_{622}^{*3} = |C_{622}^3| \hat{y} \]  

(3.47)  

(3.48)  

Since the conjugate reverses the sign of the \( \hat{x} \) component, then the terms \( B_{622}^2 \) and \( B_{622}^{*2} \) are equal in a plane symmetric system. Similarly, \( C_{622}^{*3} \), \( C_{622}^3 \), and \( C_{622}^{*3} \) are equal. Therefore, the 8th order NAT wavefront for a plane symmetric system has 12 coefficients that determine the field dependence of the Zernike FFD. It is instructive to write this out with the explicit plane-symmetric constraints shown:

\[
W_{z_{ia}} = \frac{1}{2} \left[ W_{222} + W_{422} H^2 + W_{622} H^4 - 2 \vec{H} \cdot \left( |A_{222}^*| \hat{y} + 2H^2 |A_{622}^*| \hat{y} \right) \vec{H}^2 \right.
\]
\[
+ 3H^2 \left[ \vec{B}_{422} \cdot \hat{y} + |\vec{B}_{622}^*| \left( 6H \hat{y} + \vec{H} \hat{y} \right) \right]
\]
\[
\left. - 2 \left( |A_{222}^*| \hat{y} + H^2 |A_{622}^*| \hat{y} \right) \vec{H} \right] \hat{y} \vec{H} \ldots
\]
\[
+ \left[ |C_{422}^*| \hat{y} \vec{H} + 4 |C_{622}^*| \hat{y} \vec{H} \left[ H^2 + 4 \left( \vec{H} \cdot \hat{y} \right)^2 \right] \right]
\]
\[
\left. + \left[ \vec{B}_{422}^* \hat{y} \vec{H}^* + |\vec{B}_{622}^*| \hat{y} \right] \hat{y} \vec{H} \right] \hat{y} \vec{H} \ldots
\]

(3.49)  

where we have made the following substitution:

\[
\left( H^2 \vec{H} + \vec{H}^3 + 3H^2 \vec{H} \right) = \vec{H} \left[ H^2 + 4 \left( \vec{H} \cdot \hat{y} \right)^2 \right]
\]  

(3.50)  

There are terms in Eq. (3.49) that contain a SVP operation with the unit vector \( \hat{y} \). Any SVP with the unit vector \( \hat{y} \) can be simplified with the following identity:

\[ \vec{A} \hat{y} = \vec{A} \]  

(3.51)
Therefore, the plane symmetric NAT wavefront becomes:

\[
W_{\text{z,sym}} = \frac{1}{2} \left[ W_{222} + W_{422} H^2 + W_{622} H^4 \right] \tilde{H}^2 + \ldots \\
+ 3H^2 \left[ \tilde{B}_{422} \tilde{\dot{y}} + \tilde{B}_{622} \left( 6H^2 \tilde{\dot{y}} + H^4 \right) \right] + \ldots \\
- 2A_{422} \tilde{H}^2 + 2A_{422} \left[ H^2 \tilde{H} + \left( \tilde{H} \cdot \tilde{\dot{y}} \right) \tilde{H} \right] + \ldots \\
- 3C_{422} \tilde{H}^2 + 4C_{622} \left[ H^2 \left( \tilde{H}^2 + 4 \left( \tilde{H} \cdot \tilde{\dot{y}} \right)^2 \right) \right] + \ldots \\
+ D_{622} \left( \tilde{H}^2 + \tilde{B}_{622} \tilde{\dot{y}} \right) 
\]

(3.52)

Now each term can be separated into its \( \hat{x} \) and \( \hat{y} \) components to obtain the contributions to the \( Z_5 \) and \( Z_6 \) FFDs, respectively. The field dependence of the \( Z_5 \) and \( Z_6 \) polynomials separated into each NAT term is shown in Table 3.1.

**Table 3.1** Field dependence of \( Z_5 \) and \( Z_6 \) for a y-z plane symmetric system, broken down into each NAT term and listed in ascending wavefront expansion order.

<table>
<thead>
<tr>
<th>NAT Coefficient</th>
<th>( Z_5 ) Field Dependence</th>
<th>( Z_6 ) Field Dependence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_{222} )</td>
<td>( H_y^2 - H_x^2 )</td>
<td>( 2H_y H_x )</td>
</tr>
<tr>
<td>( A_{422} )</td>
<td>(-2H_x)</td>
<td>(-2H_x)</td>
</tr>
<tr>
<td>( B_{422} )</td>
<td>( H_x^2 - H_y^2 )</td>
<td>( 2H_x H_y )</td>
</tr>
<tr>
<td>( A_{422} )</td>
<td>(-4H_y^3)</td>
<td>(-2H_x \left( H_x^2 + 3H_y^2 \right))</td>
</tr>
<tr>
<td>( B_{422} )</td>
<td>( 3H_x^2 )</td>
<td>( 0)</td>
</tr>
<tr>
<td>( C_{422} )</td>
<td>(-H_y)</td>
<td>( H_x)</td>
</tr>
<tr>
<td>( W_{622} )</td>
<td>( H_x^2 - H_y^2 )</td>
<td>( 2H_x H_y H_x^4)</td>
</tr>
<tr>
<td>( A_{622} )</td>
<td>( 2H_x \left( H_y^2 - 3H_x^2 \right) H_x^2)</td>
<td>(-2H_x \left( H_x^2 + 5H_y^2 \right) H_x^2)</td>
</tr>
<tr>
<td>( B_{622} )</td>
<td>( 7H_x^4 + 7H_x^2 ) ( + 6H_x^2 H_y^2)</td>
<td>( 4H_x H_y \left( H_y^2 - H_x^2 \right))</td>
</tr>
<tr>
<td>( C_{622} )</td>
<td>(-4H_x \left( H_x^2 + 5H_y^2 \right))</td>
<td>(-4H_x \left( H_x^2 + 5H_y^2 \right))</td>
</tr>
<tr>
<td>( D_{622} )</td>
<td>( H_x^2 - H_y^2 )</td>
<td>(-2H_x H_y)</td>
</tr>
</tbody>
</table>
The FFD of each NAT term from Table 3.1 is shown in Figure 3.1. The Z5 and Z6 components are combined and represented by a line marker. The magnitude and orientation of the line at each field point is given by Eq. (3.53). The line magnitude represents the peak-to-valley magnitude of the Z5/6 wavefront error, and the angle represents the orientation of the peaks of the wavefront error relative to the image plane.

$$|Z_{5/6}| = \sqrt{z_5^2 + z_6^2}$$

$$\phi_{Z_{5/6}} = \frac{1}{2} \tan^{-1} \left( \frac{z_6}{z_5} \right).$$

(3.53)

![Figure 3.1 The Zernike Z5/6 Full Field display of each NAT plane-symmetric field-dependence](image)

### 3.4 Estimating required surface departure

The aberration theory of freeform surfaces (ATFS) and later the work of Bauer et al. provide a framework and design method for freeform mirror designs [17,30]. We can use the methods exposed therein to theorize a calculus that allows us to quantitatively analyze...
a given design. The analysis we present here involves determining each surface’s effect on a given set of NAT aberrations. We have completed this analysis for three Zernike surface shapes: Z5 (astigmatism), Z8 (coma), and Z11 (trefoil).

### 3.4.1 Zernike Astigmatism shape (Z5)

From previous work, we know that adding a Z5 shape to any surface in the design produces field-constant astigmatism \( \bar{B}_{222} \). The overall field-constant astigmatism is given by [Eq. (14) from Ref [17]]:

\[
\vec{B}_{222} = \sum_{j=1}^{n} \mu F \overline{B}_{222,j} \ .
\]

(3.54)

Here, \( \overline{B}_{222} \) is the field-constant astigmatism from the geometry of the spherical surfaces and is proportional to the sigma vectors of the tilted surfaces. Similarly, a Z5 shape on a given surface \( j \) gives \( \mu F \overline{B}_{222,j} \), the field-constant astigmatism term for that surface. \( \mu F \overline{B}_{222,j} \) is proportional to the Z5/6 surface coefficients for that surface. In the plane symmetric case, it is proportional to Z5:

\[
\mu F \overline{B}_{222,j} = B_{222,j}^{5} Z_{s,j} \ .
\]

(3.55)

Here, we have introduced a new notation using \( \alpha \) to denote the proportionality constant \( B_{222,j}^{5} \) between a given Zernike surface shape coefficient \( Z_{s,j} \) and a given NAT coefficient for a surface \( j \). We have also dropped the vector notation for the plane symmetric case for simplicity. We see from Eqs. (3.54) and (3.55) that to completely
correct $\vec{b}_{22}^j$ using a given surface $j$, we set the surface coefficient $Z_{5,j}$ equal to the total field-constant astigmatism divided by its proportionality constant:

$$Z_{5,j}|_{B_{22}=0} = \frac{g B_{22}^j}{B_{22,j}}$$ (3.56)

Equation (3.56) assumes there is a linear relationship between the Z5 coefficient of a given surface and the NAT coefficient for field-constant astigmatism, which we implicitly assume based on [17] and which is validated by the design methods shown in [3,4,30,67].

### 3.4.2 Zernike Coma shape (Z8)

Once field-constant astigmatism is removed, often the next largest aberration is field-constant coma ($\vec{A}_{431}$) together with field-linear, field-asymmetric astigmatism ($\vec{A}_{222}$). From [17], we know that adding a Z8 shape to an optical surface produces both field-linear, field-asymmetric astigmatism ($\vec{A}_{222}$) and field-constant coma ($\vec{A}_{431}$) terms as well as field-linear defocus ($\vec{A}_{220M}$), also known as focal plane tilt [17]:

$$\vec{A}_{431} = g \vec{A}_{431} - \sum_{j=1}^{n} f_{FP} \vec{A}_{431,j}$$ (3.57)

$$\vec{A}_{222} = g \vec{A}_{222} - \sum_{j=1}^{n} \left( \frac{x}{y} \right)_{FP} \vec{A}_{431,j}$$ (3.58)

$$\vec{A}_{220M} = g \vec{A}_{220M} - \sum_{j=1}^{n} \left( \frac{x}{y} \right)_{FP} \vec{A}_{431,j}$$ (3.59)

We can again write the proportionality constants for these terms relative to the surface coefficient Z8 (and again dropping the vector notation):
Here, we will show an example of how to correct field-asymmetric, field-linear astigmatism ($\tilde{A}_{22}$) and field-constant coma ($\tilde{A}_{31}$) using two surfaces. The residual blur from the focal plane tilt can be corrected by tilting the physical image plane, so its proportionality constant has not been defined here.

Section 3.3.1 showed how to determine $A_{22}$. Field-constant coma, $A_{31}$, is equivalent to the Zernike coma (Z8) coefficient at the central field point e.g. $(H_x, H_y) = (0,0)$. To simultaneously correct both $\tilde{A}_{31}$ and $\tilde{A}_{22}$, we know from [30] that we need two freeform surfaces with a Z8 surface shape. The total magnitude of these two aberrations in a system with two surfaces using a Z8 shape is given by combining Eqs. (3.57) and (3.58) with Eqs. (3.60) and (3.61):

$$A_{31} = A_{31,g} + A_{31,\alpha}z_{4,\alpha} + A_{31,\beta}z_{4,\beta} = 0$$

$$A_{22} = A_{22,g} + A_{22,\alpha}z_{4,\alpha} + A_{22,\beta}z_{4,\beta} = 0$$

Solving this system of equations for the required surface coefficients gives

$$z_{8,\alpha} \bigg|_{A_{31}, A_{22} = 0} = -\frac{A_{22,\alpha}^u \left( g A_{31} \right) + A_{22,\beta}^u \left( g A_{22} \right)}{A_{31,\beta}^u A_{22,\beta}^u - A_{31,\beta}^u A_{22,\alpha}^u}$$

$$z_{8,\beta} \bigg|_{A_{31}, A_{22} = 0} = \frac{A_{22,\alpha}^u \left( g A_{31} \right) - A_{31,\alpha}^u \left( g A_{22} \right)}{A_{31,\beta}^u A_{22,\beta}^u - A_{31,\beta}^u A_{22,\alpha}^u}$$
It is convenient to define the ratio of the $A_{131}$ to $A_{222}$ proportionality constants for each surface and for the ratio of the overall aberrations:

$$R_{z_e,j} \equiv \frac{A_{222,j}^a}{A_{131,j}^a}$$

$$R_{z_e,sys} = \frac{g A_{222}}{g A_{311}}$$

Rewriting Eq. (3.64), we get the solution in terms of these ratios:

$$z_{8,a} \bigg|_{A_{131},A_{222}=0} = \left(\frac{q A_{311}^a}{A_{131,b}^a} \right) \left( \frac{R_{z_e,sys} - R_{z_e,b}}{R_{z_e,a} - R_{z_e,b}} \right)$$

$$z_{8,b} \bigg|_{A_{131},A_{222}=0} = \left(\frac{q A_{311}}{A_{311,b}} \right) \left( \frac{R_{z_e,sys} - R_{z_e,a}}{R_{z_e,a} - R_{z_e,b}} \right)$$

One way to understand Eq. (3.66) is to imagine incrementally changing the coefficient for a given surface, say $z_{8,a}$, until the ratio of the overall aberrations $z_{sys}/A_{311}$ is equal to the ratio of the proportionality constants for the other surface coefficient, $z_{8,b}$. Then, we can increment that surface coefficient $z_{8,b}$ until both aberrations vanish. Eq. (3.66) accomplishes this task by solving the system of equations directly.

### 3.4.3 Zernike Trefoil shape (Z11)

After correcting $\tilde{A}_{222}, \bar{A}_{222}, \bar{A}_{131}$, often the next largest non-symmetric aberrations are field-constant elliptical coma $\bar{C}_{333}^1$ and field-linear, field-conjugate astigmatism $\bar{C}_{222}^1$. From [17] and [30], we know that a Zernike trefoil (Z11) shape on a surface produces these aberrations, and therefore we can use a similar procedure as in Section 3.4.2 to obtain
the required Z11 surface coefficients for two surfaces to simultaneously correct $\tilde{C}_{333}^3$ and $\tilde{C}_{422}^3$.

\begin{equation}
\begin{aligned}
z_{11,a} \bigg|_{c_{333,422}^2=0} &= \frac{\left( C_{333,6}^1 \right) \left( R_{Z_{11,a}} - R_{Z_{11,b}} \right)}{C_{333,6}^1 \left( R_{Z_{11,a}} - R_{Z_{11,b}} \right)} \\
z_{11,b} \bigg|_{c_{333,422}^2=0} &= \frac{\left( C_{333,6}^1 \right) \left( R_{Z_{11,a}} - R_{Z_{11,b}} \right)}{C_{333,6}^1 \left( R_{Z_{11,a}} - R_{Z_{11,b}} \right)} 
\end{aligned}
\end{equation}

(3.67)

### 3.5 Design Example

The 8th order NAT expansion can be used to estimate the required freeform departure to correct various NAT aberrations for a given freeform design, and significantly, this is done without ray trace optimization. Using the field dependences in Table 3.1, programs were implemented in Code V to compute the NAT coefficients given the $Z_5$ and $Z_6$ wavefront coefficients at field points throughout the FOV. To estimate the NAT coefficients, the Code V algorithm uses Singular Value Decomposition to solve the system of equations given by Table 3.1 with the input of the $Z_5$ and $Z_6$ FFD data. As an example, we show the use of this method using a design from the literature.

The example design is a freeform three mirror compact (TMC) design used in [30]. This design is the classic TMC geometry made more compact using freeform surfaces. It uses a positive-negative-positive optical power distribution on the mirrors and has the stop at the primary. All analyses are done at a wavelength of 587 nm. A cross-sectional layout of the design stripped of the freeform terms, leaving only the base spherical surfaces, is shown in Figure 3.2.
The process of adding freeform shapes to the surfaces starts with an all-spherical design. We started with the final design from [30] and removed the freeform terms, leaving only the tilted spherical surfaces. The first step towards adding freeform is to examine the FFDs and the relevant NAT aberration coefficients. The FFDs for the design with only spherical surfaces are shown in Figure 3.3. Select NAT aberration coefficients derived from these FFDs are shown in Figure 3.4.

From this point, we would like to estimate the required coefficients to remove, in order, $B_{222}$, $A_{222}/A_{131}$, and $C_{422}/C_{333}$ using Eqs. (3.56), (3.66), and (3.67), respectively. However, first we must calculate the respective aberration ratios and therefore the respective proportionality constants. One way to calculate these values is to use the relations given by Fuerschbach et al. that relate the Zernike surface shape coefficients to the NAT aberration terms [17]. In practice, however, these relationships are difficult to implement and produce a small inaccuracy because they do not consider the induced aberrations of the system. In this example, we have opted to use a differential technique to estimate the proportionality constants. To do so, the NAT coefficients are first estimated.
Then a single Zernike surface coefficient is changed by a small amount, and the NAT coefficients are estimated again. This technique has the advantage of including any induced aberration effects. Additionally, it directly relates the coefficient in the design software with the aberration, regardless of the Zernike normalization radius.

Figure 3.3. Full field displays for the design with all spherical surfaces. Units are waves at 532 nm. (a) RMS WFE (b) Defocus (Z4) (c) Astigmatism, Z5/6 (d) Coma, Z7/8 (e) Spherical, Z9 (f) Elliptical Coma, Z10/11

3.5.1 Correcting $B_{222}^2$

From Figure 3.3 we can see that the dominant aberration is primarily field-constant astigmatism, so we begin by correcting field-constant astigmatism as in Bauer et al. [30]. However, here we make use of our ability to estimate $B_{222}$. Table 3.2 shows the result of two iterations of this process. Using Eq. (3.56), we can predict that adding -0.0234 waves of Z5 to the stop surface will remove this field-constant astigmatism, and after it is added,
we see that the $B_{222}$ term is reduced by a factor of $7e4$. A second iteration of re-estimating the proportionality constant and adjusting the surface coefficient accordingly further reduces $B_{222}$ to negligible levels. The resulting Z5 FFD is shown in Figure 3.6b.

### Table 3.2. $B_{222}$ removal process with two iterations

<table>
<thead>
<tr>
<th></th>
<th>Iteration 1</th>
<th>Iteration 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Starting $B_{222}$ Value (waves)</td>
<td>130.911</td>
<td>-0.018</td>
</tr>
<tr>
<td>$B_{222,M1}$ (waves/micron)</td>
<td>5.59</td>
<td>5.63</td>
</tr>
<tr>
<td>$Z_{5,M1}</td>
<td><em>{B</em>{222}=0}$ (microns)</td>
<td>-23.4261</td>
</tr>
<tr>
<td>Final $B_{222}$ Value (waves)</td>
<td>-0.018</td>
<td>8.9e-008</td>
</tr>
</tbody>
</table>

Figure 3.4. Selected NAT coefficients related to the Zernike surface terms $Z_5$ (astigmatism), $Z_8$ (coma), and $Z_{11}$ (trefoil, or elliptical coma) shown at each step.

Figure 3.5. (a) The ratios of the $A_{222}$ and $A_{131}$ from adding a $Z_8$ shape onto each mirror and for the overall aberrations seen in Figure 3.4. (b) The ratios of the $C_{422}$ and $C_{333}$ from adding a $Z_{11}$ shape onto each mirror and for the overall aberrations seen in Figure 3.4.
3.5.2 Correcting $A_{222}$ and $A_{131}$ simultaneously

Once $B_{222}$ is removed, $A_{222}$ and $A_{131}$ are the next largest aberrations, as can be seen in Figure 3.4. Using Eq. (3.66), we can estimate the coefficients required for any pair of surfaces e.g. M1 & M2 or M1 & M3, etc. Examining the ratios in Figure 3.5a can help determine which two surfaces will be most effective in removing these two aberrations. The “effectiveness” is based on the required departure that each surface needs to remove these two aberrations. In this particular example, we see that the Z8 ratio for the system as a whole is smaller than the ratios for M2 and M3. M1, the stop surface, has a ratio that is zero, since a Z8 shape at the stop only produces field-constant coma in the wavefront. Therefore, just by looking at the ratios, we know that by adding some amount of Z8 to the stop surface, we can increase the magnitude of the overall system ratio until it is equal to the ratio of M2. From there, adding any amount of M2 will either increase or decrease both $A_{222}$ and $A_{131}$ the same relative amount, and we can then drive both terms to zero. The same could be done for M2 and M3, but using M2 to make the system ratio equal to the M3 ratio would require more surface aberrations to be added and therefore more surface departure.

This analysis can be summarized by Table 3.3. Using M1 and M2 to remove $A_{222}$ and $A_{131}$, we can see the resulting Z5/6 FFD in Figure 3.6c and the resulting Z7/8 FFD in Figure 3.7c.
Table 3.3. \( A_{222}/A_{131} \) ratio for each pair of surfaces and the resulting predicted surface coefficients

<table>
<thead>
<tr>
<th>Overall Z8 aberrations (waves)</th>
<th>Z8 proportionality constants and ratio values (waves/micron)</th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g \cdot A_{222} )</td>
<td>( A_{222,j}^i )</td>
<td>-0.02141</td>
<td>-1.649</td>
<td>1.491</td>
</tr>
<tr>
<td>( g \cdot A_{131} )</td>
<td>( A_{131,j}^i )</td>
<td>-3.019</td>
<td>1.267</td>
<td>-0.4658</td>
</tr>
<tr>
<td>Ratio</td>
<td>Ratio</td>
<td>0.00709</td>
<td>-1.30</td>
<td>-3.20</td>
</tr>
</tbody>
</table>

Predicted Surface Coefficients (microns)

<table>
<thead>
<tr>
<th>M1 + M2</th>
<th>M2 + M3</th>
<th>M1 + M3</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.59</td>
<td>-38.48</td>
<td>-</td>
</tr>
<tr>
<td>-</td>
<td>-56.95</td>
<td>-20.50</td>
</tr>
<tr>
<td>14.16</td>
<td>-</td>
<td>42.69</td>
</tr>
</tbody>
</table>

3.5.3 Correcting \( C_{333}^3 \) and \( C_{422}^4 \) simultaneously with a Z11 shape on two mirrors

The relevant proportionality constants and ratios can be similarly analyzed for a Z11 shape on two surfaces. The relative ratios for each mirror and for the overall aberrations are shown in Figure 3.5b. In the case of a Z11 shape, both non-stop surfaces have positive ratios along with the overall system ratio. The M3 ratio is closest to the overall aberration ratio.

Using a Z11 shape on M1 will add only field-constant elliptical coma (\( C_{333}^3 \)), which will lower the \( R_{Z1,sys} \) ratio. If the correct amount is added, the \( R_{Z1,sys} \) ratio will be equal to the M3 ratio. Then, adding a Z11 shape on M3 will reduce both \( C_{333} \) and \( C_{422} \) in the right proportions to simultaneously drive them both to zero. We could instead add even more \( C_{333}^3 \) to lower the overall \( R_{Z1,sys} \) ratio further to equal the M2 ratio, but this would require adding more surface departure and more aberrations than in the M1-M3 case. Similarly, the combination of M2 and M3 could be used. However, this would be even less efficient than either of the other two cases, since more departure and aberrations would need to be added using M2 to change the overall aberration ratio to the ratio of M3. Therefore, we proceed with M1 and M3 and use Eq. (3.67) to remove \( C_{422} \) and \( C_{333} \) simultaneously. The
resulting Z5/6 FFD is shown in Figure 3.6d and the resulting Zernike Elliptical Coma (Z10/11) is shown in Figure 3.8d. In this case, only a single iteration was necessary to reduce both $C_{422}$ and $C_{333}$ below 1e-7 waves.

Figure 3.6. Astigmatism FFDs after each step. The axes units are the object field angle coordinates in degrees. (a) Starting design without any freeform (spheres only). (b) After $B_{222}$ removal. (c) After $A_{222}/A_{131}$ removal. (d) After $C_{422}/C_{333}$ removal. (e) After removing the residuals of each terms through another iteration of each step.

Figure 3.7. Zernike Coma (Z7/8) FFDs for each step. The axes units are the object field angle coordinates in degrees. (a) Starting design without any freeform (spheres only). (b) After $B_{222}$ removal. (c) After $A_{222}/A_{131}$ removal. (d) After $C_{422}/C_{333}$ removal. (e) After removing the residuals through another iteration of each step.

Figure 3.8. Zernike Elliptical Coma (Z10/11) for each step. The axes units are the object field angle coordinates in degrees. (a) Starting design without any freeform (spheres only). (b) After $B_{222}$ removal. (c) After $A_{222}/A_{131}$ removal. (d) After $C_{422}/C_{333}$ removal. (e) After removing the residuals through another iteration of each step.

3.5.4 Discussion

Examining Figure 3.4 and Figure 3.6d, we see that there is some residual field-constant astigmatism ($B_{222}$) that has been accrued in the $A_{222}/A_{131}$ correction step even though it
was already corrected using a Z5 shape at M1, as indicated by the non-zero Z5/6 seen at the center of the FFD in Figure 3.6c. The exact cause of this is not obvious by considering only the ATFS and NAT alone, but induced aberrations are likely to blame [68]. Once we begin adding many waves of, for instance, Z8 departure onto surfaces that have non-collimated beam footprints as in Section 3.5.2, the assumptions that allow us to ignore induced aberrations begin to break down. However, applying another iteration of each step removes the small residual \(B_{222}, A_{222}/A_{131},\) and \(C_{422}/C_{333}\) aberrations. The resulting FFDs are shown in Figure 3.6e, Figure 3.7e, and Figure 3.8e. Additionally, Figure 3.4 shows the relevant NAT coefficients at each step and how each step affects the residual aberrations of the others.

### 3.5.5 Higher order Zernike surface coefficients

The ATFS as detailed in [17] includes analysis of higher-order Zernike terms beyond Z10/11. Additionally, previous work detailing a freeform design method that relies on the ATFS includes a qualitative analysis of the FFDs for these higher-order Zernike surface shapes [30]. In the present work, we have carried out a quantitative analysis only up to Z10/11 surface shapes because shapes with higher-order than this produce more complicated field-dependence in Zernike aberration FFDs besides astigmatism (Z5/6). Using similar methods for the higher-order FFDs is certainly feasible, but it requires analyzing the field-dependence of the relevant Zernike aberrations, which has not yet been completed.
3.6 Conclusion

We have expanded the field-dependence of Zernike astigmatism (Z5/6) up to 8th order in wavefront by expanding the Nodal Aberration Theory wavefront up to 8th order and collecting the terms with new field dependence. We have then shown how to use insights from the ATFS to estimate the required Zernike surface coefficients to correct certain NAT terms in plane-symmetric optical systems as a tool for freeform design. The Z5, Z8, and Z11 surface shapes for plane-symmetric optical systems have been analyzed and methods to predict their required coefficients given. As an example, a three-mirror freeform design from previous work was analyzed using these methods and the method was shown to be effective at predicting the required surface coefficients and correcting the intended aberrations.

3.7 Chapter 3 Appendix

Useful Shack Vector Product identities:

\[
(h \cdot \sigma) (\bar{h} \cdot \sigma) \cdot \rho^2 = \frac{1}{2} H^2 \sigma_j \cdot \rho^2 + \frac{1}{2} \sigma_j \bar{H}^2 \cdot \rho^2, \tag{3.68}
\]

\[
(h \cdot \sigma_j^*) (\bar{h} \cdot \rho^2) = \frac{1}{2} H^4 \sigma_j \cdot \rho^2 + \frac{1}{2} \bar{H}^4 (\sigma_j^*)^2 \cdot \rho^2, \tag{3.69}
\]

\[
2(h \cdot \sigma_j)(\sigma_j^* \cdot \rho^2) = \sigma_j^2 \sigma_j \bar{H} \cdot \rho^2 + \sigma_j \bar{H} \rho^2 = \sigma_j^2 \sigma_j \bar{H} \cdot \rho^2 + \sigma_j \bar{H} \cdot \rho^2, \tag{3.70}
\]

\[
2(\bar{A}^2 \cdot \bar{B}^2)(\bar{A} \cdot \zeta \bar{C}^2) = (\bar{A} \cdot \bar{A})(\bar{B} \cdot \zeta \bar{C}^2) + (\bar{B} \cdot \bar{B})(\bar{A} \cdot \zeta \bar{C}^2), \tag{3.71}
\]

\[
2(\bar{h} \cdot \sigma_j)(\bar{h} \cdot \sigma \cdot \rho^2) = H^2 \sigma_j \cdot \bar{H} \cdot \rho^2 + \sigma_j^2 \bar{H} \cdot \rho^2 \tag{3.72}
\]

\[
2\sigma_j \cdot (h \cdot \sigma_j)(\sigma_j^* \cdot \rho^2) = H \sigma_j^2 (\bar{h} \cdot \rho) + H^2 (\bar{h} \cdot \sigma_j) \cdot \rho^2, \tag{3.73}
\]
\[ 2\sigma_i^2 (\bar{H} \cdot \bar{\sigma}_j) (\bar{H}^2 \cdot \bar{\rho}^2) = H^2 \sigma_i^2 (\bar{\sigma}_j \bar{H} \cdot \bar{\rho}^2) + \sigma_i^4 (\bar{\sigma}_j \bar{H}^2 \cdot \bar{\rho}^2), \quad (3.74) \]

\[ (\bar{H}^2 \cdot \bar{\sigma}_j) (\bar{\sigma}_j^2 \cdot \bar{\rho}^2) = \frac{1}{2} (\bar{\sigma}_j^2 \cdot \bar{\sigma}_j) (\bar{H}^2 \cdot \bar{\rho}^2) + \frac{1}{2} \bar{\sigma}_j^4 \cdot \bar{H}^2 \bar{\rho}^2 
\quad = \frac{1}{2} \sigma_i^4 (\bar{H}^2 \cdot \bar{\rho}^2) + \frac{1}{2} (\bar{\sigma}_j \bar{H}^2 \cdot \bar{\rho}^2), \quad (3.75) \]

\[ 4\sigma_i^2 (\bar{H} \cdot \bar{\sigma}_j) (\bar{H} \bar{\sigma}_j \cdot \bar{\rho}^2) = 2\sigma_i^2 H^2 (\bar{\sigma}_j^2 \cdot \bar{\rho}^2) + 2\sigma_i^4 (\bar{H}^2 \cdot \bar{\rho}^2), \quad (3.76) \]

\[ 2\sigma_i^4 (\bar{H} \cdot \bar{\sigma}_j) (\bar{\sigma}_j^2 \cdot \bar{\rho}^2) = \sigma_i^4 (\bar{\sigma}_j \bar{H} \cdot \bar{\rho}^2) + \sigma_i^2 (\bar{\sigma}_j \bar{H} \cdot \bar{\rho}^2). \quad (3.77) \]
4 Volume comparison of unobscured TMC designs

One of the advantages often ascribed to freeform optical surfaces is their ability to reduce the mass or volume of rotationally non-symmetric optical systems while maintaining (and in some cases, exceeding) optical performance [5,15,30,31]. To investigate this volume reduction capability of freeform surfaces, we focus on the design of unobscured three mirror imagers. There are many traditional forms of such optical systems, including the three-mirror anastigmat (TMA) and the reflective triplet. Also known as the three-mirror compact (TMC), the unobscured reflective triplet is often touted as the most compact (in terms of overall length and volume) form of unobscured three mirror imager design [44,69,70]. The TMC and TMA have been popular design forms for improvement using freeform surfaces [31–34].

The present work compares the effectiveness of two surface types in reducing the volume of unobscured TMC designs. One is a traditional surface type, the off-axis section of a rotationally symmetric aspheric parent. The second surface type is a field-centered FRINGE Zernike freeform surface. We will use the FRINGE ordering of terms as specified in the CODE V manual throughout this chapter, so “FRINGE Zernike” will be shortened to simply “Zernike” or interchangeably “freeform” throughout [71]. We used Zernike polynomial surfaces in this study for a number of reasons. First, they are a complete polynomial set and they are orthogonal. Second, they are the basis set used in the aberration theory of freeform surfaces (ATFS) [17]. Additionally, Zernikes are well understood, simple to implement, and typically provide the required degrees of freedom according to
the ATFS. Others have shown that Zernikes are equally capable for freeform design as other orthogonal polynomials [72], and there are many examples of reflective systems using Zernike polynomial surface descriptions [17,30,73,74].

The designs using sections of rotationally symmetric parent surfaces will be referred to as three-mirror compact asphere (TMCA) designs and the designs using Zernike surfaces will be referred to as three-mirror compact freeform (TMCF) designs in this work.

In systems using off-axis sections of rotationally symmetric surfaces, field-bias and aperture offset are used to unobscure the system. The field bias and aperture offset are akin to a tilt and/or decenter of the surface with respect to the local object and image and the local entrance and exit pupils for that surface. Because the field bias and aperture offset remove any rotational symmetry in the field-dependence of the aberrations, the parent surfaces of the TMCA designs should not be restricted to sharing an axis of rotational symmetry, as this restriction would impose an added constraint on the TMCA designs not present in the TMCF designs. This is analogous to the TMCF, which uses tilted surfaces to avoid obscuration rather than field-bias and aperture offset. Thus, the only difference from an aberration standpoint between a non-coaxial TMCA and a TMCF are the restrictions on the surface shape that can be used to correct the optical aberrations. Freeform surfaces allow degrees of freedom that rotationally symmetric surfaces do not allow because their aberration contributions can be decoupled from each other, as shown by Fuerschbach et al [29].
The basic thesis of this work is that non-coaxial TMCAs have fewer layout options than TMCFs. The way we exemplify this property is by starting with plane-symmetric, unobscured non-coaxial TMCA designs from the literature and showing that, by adding freeform terms to the surfaces, more plane-symmetric aberrations can be corrected; thus, allowing more layout options for a given performance level, or better performance for a given first-order layout.

First, we describe the design process for the TMCA and a process to progressively reduce the volume target. Next, we show that these TMCA designs can be converted to TMCF designs with the same surface shapes but described using centered Zernike polynomials, which we term the TMCF Converted designs. We then show that, without changing the first-order layout, the Zernike polynomials can further correct aberrations that were not possible with the off-axis asphere descriptions of the TMCA. These designs will be termed TMCF Frozen-Geometry designs. Next, we use the same volume reduction algorithm on the TMCF designs to progressively reduce the volume to determine the volume-performance relationship for comparison to the TMCA-based designs. These designs are termed TMCF Volume-Optimized designs. Finally, we summarize and discuss the results by looking at the aberration full-field displays (FFDs) and the surface departures. For readability, the acronyms and descriptions for each design completed in this work are summarized in Table 4.1.
Table 4.1. Naming convention and descriptions for the designs completed in this work

<table>
<thead>
<tr>
<th>Design Name</th>
<th>TMCA</th>
<th>TMCF Converted</th>
<th>TMCF Frozen-Geometry</th>
<th>TMCF Volume-Optimized</th>
</tr>
</thead>
<tbody>
<tr>
<td>Surface Type</td>
<td>Off-axis Q&lt;sub&gt;CON&lt;/sub&gt;</td>
<td>Centered Fringe Zernike</td>
<td>Centered Fringe Zernike</td>
<td>Centered Fringe Zernike</td>
</tr>
<tr>
<td>Description</td>
<td>Off-axis asphere designs, optimized for a given volume from 110 L to 70 L</td>
<td>Off-axis asphere designs converted to centered Zernike surface types (Identical in performance and first-order layout to TMCA designs)</td>
<td>TMCF Converted designs whose surface shapes are optimized for aberration correction</td>
<td>TMCF designs optimized for a given volume from 110 L to 50 L</td>
</tr>
<tr>
<td>Layout optimized for a given volume?</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Surface coefficients optimized?</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

4.1 Traditional TMC design – the TMCA

The TMC uses a positive-negative-positive power distribution without an internal image, similar to the (refractive) Cooke triplet. Traditionally, TMCs are effective with the aperture stop at the primary or at the secondary [44], both having certain advantages depending on the use case. The aperture stop at the secondary is best for controlling aberrations (similar to the Cooke triplet) and allows for telecentricity in image space if needed. This work will focus on TMC configurations with the aperture stop at the secondary.

Traditional design techniques for unobscured three mirror imagers using rotationally symmetric surfaces typically start with a co-axial, rotationally symmetric, third-order aberration corrected design, which can be obtained using myriad methods [75–77]. These co-axial designs then use field-bias or aperture offset to remove the obscuration. Many designs restrict the parent surfaces to remain co-axial, while others allow the parent surfaces to tilt and decenter relative to one another after they are unobscured. Other approaches do not start from rotationally symmetric designs, but instead consider tilted
components with rotationally symmetric parent surfaces (usually beginning with off-axis conics) [19, 42, 52, 53, 78]. These techniques may start with given airspaces and radii, but do not allow specification of volume as a parameter to constrain the solution. One difficulty with specifying the volume as a constraint on the first- or third-order solution from these techniques is that there may be many first- or third-order layouts that lead to a given volume from these techniques. In this study, since we are primarily concerned with the relationship between volume and performance, we started from a well-corrected non-coaxial TMCA design example and evolved it to meet our performance goals. The alternative would have been to create a new first- or third-order starting point at each volume target.

![Figure 4.1. The layout of the thremrc.len example lens in CODE V, a TMCA type design.](image)

The reference design is the *thremrc.len* file included with the CODE V optical design software shown in Figure 4.1. The reference design uses a conic primary, a conic secondary, and a 10th order aspheric tertiary. This design was adapted to first fit the system specifications used in this work, as reported in Table 2. Specifically, the system was scaled up by 2.5x to increase the entrance pupil diameter (EPD) from 100 mm to 250 mm. Next, all three surfaces were converted to Q_{CON} surface types, which allow up to 30th radial order aspheric surface sag to be added to a base conic [79]. Using aspheres with higher-order
polynomial terms allows for more aberration correction when the design is re-configured for a smaller volume than a simple conic surface might allow. Q\textsubscript{CON} surfaces are normalized and orthogonal, which helps with convergence, as shown by Forbes [79].

Table 4.2. System specifications for the TMCA and TMCF designs

<table>
<thead>
<tr>
<th>Parameter</th>
<th>TMCA Specification</th>
<th>TMCF Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entrance pupil diameter (mm)</td>
<td>250</td>
<td>Same</td>
</tr>
<tr>
<td>F-Number</td>
<td>F/3.0</td>
<td>Same</td>
</tr>
<tr>
<td>Number of mirrors</td>
<td>3</td>
<td>Same</td>
</tr>
<tr>
<td>Distortion</td>
<td>&lt; 5%</td>
<td>Same</td>
</tr>
<tr>
<td>Square field-of-view (degrees)</td>
<td>3.75 x 3.75</td>
<td>Same</td>
</tr>
<tr>
<td>Surface type</td>
<td>Off-axis Q\textsubscript{CON}</td>
<td>Centered FRINGE Zernike</td>
</tr>
<tr>
<td>Wavelength for evaluation (nm)</td>
<td>587</td>
<td>Same</td>
</tr>
<tr>
<td>Volume (L)</td>
<td>110 to 70</td>
<td>110 to 50</td>
</tr>
</tbody>
</table>

To reduce the volume, the airspaces and radii were allowed to vary so that a volume reduction constraint could be applied. The Q\textsubscript{CON} surface terms up to 22\textsuperscript{nd} order were allowed to vary. In preliminary studies, terms higher than 22\textsuperscript{nd} order did not have a beneficial effect on aberration correction and tended to slow down the optimization. The surface y-decenters and tilts about the x-axis were also allowed to vary. It was important to vary the surface decenters as it allowed different portions of the aspheric surface to be used for aberration control and was key to volume minimization. Clearance constraints were used to maintain the unobscured form. To maintain clearance, the clear-aperture of each surface is constrained to be greater than 5 mm from the closest ray.

The volume was determined by a simple algorithm calculating the smallest rectangular box that bounds all the surface-ray intersections. The surface-ray intersections were calculated in global coordinates relative to a given surface’s coordinate axes orientation. To compute the volume of the bounding box, the maximum and minimum coordinate along each (x,y,z) axis is computed and the difference is taken to give an extent along each axis.
(Lx, Ly, Lz). The volume of the bounding box is then the product of the lengths. To find the minimum bounding box, this procedure is repeated while rotating the coordinate frame, as illustrated in Figure 4.2, and the minimum volume is taken to give the smallest bounding rectangular box’s volume. This was repeated relative to each surface and the smallest volume was chosen. This simple volume algorithm allows us to quickly compute the volume and constrain the layout to a given volume target. It is important to note, however, that this volume may not be the as-built volume, for which other parameters must be considered, such as the thickness of the mirror substrates, the mirror mounting hardware, the detector housing size, etc. Even so, the optical ray-based volume we are using in this study and the as-built volume that considers these parameters are likely closely related by a multiplicative factor, with all other conditions kept the same.

By using this volume algorithm as a constraint in optimization, the radii and airspaces were able to change to fit the volume target. The volume was optimized to reduce the volume to a target of 100 L, which was chosen based on previous studies to achieve a compact, diffraction-limited design. To ascertain the performance at increasingly smaller volumes, the volume constraint target was progressively reduced in 0.1 L increments from 100 L down to 70 L to obtain TMCA designs of varying performance and volume. The
volume target was also increased from 100 L up to 110 L, using the same procedure, to see the trend. This iterative approach allowed the surface coefficients to re-optimize for each slightly smaller volume. At each volume increment, first STEP optimization was used and then damped least squares optimization was used in succession to minimize the wavefront error while maintaining the volume target (see Chapter 3 in the CODE V Optimization Reference Manual [80]). One advantage of this optimization approach is that it is able to find well-optimized designs for a given volume for both TMCA and TMCF type designs by using small volume increments and taking advantage of standard optimization routines in CODE V. One disadvantage is that this process is lengthy due to the small volume increments. It is possible that local minima are found (as with any complex optical design problem) but using STEP optimization and small increments help to avoid local minima.

The resulting wavefront error (WFE) performance ranges from a field-averaged 0.062 \( \lambda \) RMS WFE at 110 L up to 0.31 \( \lambda \) RMS WFE at 70 L. The full volume data are shown in Figure 4.3. The smallest volume for which the TMCA design has average RMS WFE at or below the diffraction limit (i.e. 0.07 \( \lambda \)) is 96.9 L.
Figure 4.3. (a) Field-averaged RMS WFE performance versus volume for each design type. The original TMCA design optimized for volume (blue line) crosses the 0.07 waves line at 96.9 L. The TMCF Frozen-Geometry design (red line) crosses the 0.07 waves line at 72.6 L. The TMCF Volume-Optimized design (yellow line) crosses the 0.07 wave line at 59.0 L. The reverse optimization from 59 L to 110 L of the TMCF Volume-Optimized design (dashed green line) avoids the local minimum of the TMCF Volume-Optimized forward optimization. (b) The same data as the chart in (a), showing more detail in the 0.0 to 0.1 waves range.

4.2 Conversion of TMCA designs to centered Zernike surfaces

To determine the amount of improvement afforded by freeform surfaces for a given first-order geometry, each TMCA design was first converted from the off-axis asphere geometry to the centered Zernike geometry. An algorithm was devised and implemented in a CODE V script to convert the TMCA designs to the equivalent centered Zernike designs. The algorithm captures a 3D surface profile of the effective aperture of each surface to produce a point cloud and the coordinates are transformed such that the chief ray of the central field point defines the origin of each surface. The coordinates are then tilted about the local x-axis such that the surface normal is perpendicular to the local X-Y plane. This configuration is conducive to using a decenter-and-bend (BEN) surface type in CODE V because there will be zero tilt at the center of the surface, and as such the tilt about the x-axis (alpha tilt in CODE V) is equal to the angle of incidence of the optical axis ray. The point cloud data for each surface is then fitted with a best fit sphere (BFS), and
the residual sag after subtraction of the BFS is fitted with Zernike terms up to Z37. For each TMCA design, the decenter-and-return (DAR) surface decenters are converted to centered decenter-and-bend (BEN) surface decenters with only a tilt about the x-axis. The surfaces are converted to Zernike FRINGE (ZFR) surface types and the computed coefficients are entered. The RMS WFE of the converted designs differ by less than 1% from 110 L to 80 L, by less than 3.5% from 80 L to 70 L, and by 0.7% on average to the original TMCA designs. These designs, referred to as “TMCF Converted” designs, as reported in Table 1, each have the same first-order geometry as the TMCA designs of the same volume, but are simply represented by centered Zernike surface types instead of off-axis aspheres.

4.3 Additional WFE correction for the TMCF Converted designs using freeform surfaces

To determine the amount of improvement afforded by freeform surfaces, the surface shapes of the TMCF Converted designs were optimized. Specifically, the Y-Z plane symmetric surface terms up to Z25 were allowed to vary. In these designs, referred to as TMCF Frozen-Geometry as reported in Table 1, the first-order geometry, as their names indicate, was kept frozen (air-spaces, radii, and surface tilt angles) to facilitate a comparison to the originating TMCA design of the same volume.

As seen in Figure 4.3, the TMCF Frozen-Geometry designs significantly improved upon the average RMS WFE of the designs. The improvement in RMS WFE compared to the TMCA designs at each volume ranges from 0.031 waves at a volume of 110 L (a 50.4%
improvement) up to 0.22 waves at a volume of 70 L (a 69.2% improvement). The average percent RMS WFE improvement over the range of volumes from 110 L to 70 L is 57.4%.

This analysis shows that, for a TMCA geometry optimized for a given volume, the performance can typically be significantly improved by allowing the surface to depart from rotational symmetry by adding freeform terms to the surface. Said another way, the optimal surfaces for an unobscured TMC with a given volume are not, in general, sections of surfaces with rotationally symmetric parents.

4.4 Additional volume reduction using freeform surfaces

We have seen that a given TMCA first-order layout can achieve a lower RMS WFE by allowing the surfaces to break rotational symmetry. However, for a given volume target, we hypothesize that the optimal TMCF first-order geometry may be different than the optimal TMCA layout, since the freeform surfaces of the TMCF can better correct the plane-symmetric aberrations. To test this hypothesis, we used the same volume reduction algorithm as the TMCA designs to create TMCF designs for each volume, starting with the 110 L TMCF Frozen-Geometry, and progressively reducing the volume target down to 50 L.

Like the TMCA volume reduction, the TMCF airspaces and radii were allowed to vary. The plane-symmetric Zernike terms up to Z25 (12 terms in total) were also allowed to vary. The magnitude of the Zernike power term and the magnitude of the linear tilt terms were constrained such that there is no Zernike power, tilt, or piston at the center of the surface. The resulting volume vs. average RMS WFE performance is shown in Figure 4.3.
The resulting design with the smallest volume having an average RMS WFE close to 0.07 waves has a volume of 59 L, a 39% reduction in volume compared to the diffraction-limited TMCA design.

In addition to the volume reduction curve for the TMCF Volume-Optimized designs shown in Figure 4.3, the optimization procedure was repeated in reverse starting from 59 L up to 110 L. As we can see, this reverse optimization produces a smoothly varying line from 80 L down to 65 L and avoids the local minimum in the forward curve.

4.5 Discussion

The above results illustrate that rotationally symmetric surfaces do not typically produce the optimal surface shapes for a given first-order geometry when used in the context of a system that inherently lacks rotational symmetry such as the unobscured TMC design form. To understand why this is the case, we compare the three design types, TMCA, TMCF Frozen-Geometry, and TMCF Volume-Optimized across similar volumes (iso-volume) and across similar performance levels (iso-performance).

4.5.1 Iso-volume comparison

It is instructive to compare the three designs types at the same volume. A volume of 72.5 L is the volume at which the TMCF Frozen-Geometry design is just below 0.07 waves average RMS WFE. At this volume, Figure 4.4 shows the layouts for each of the three design types. The layouts for both the TMCA and the TMCF Frozen-Geometry designs are different from the TMCF Volume-Optimized design. The centered Zernike surfaces are able to correct plane-symmetric aberrations according to the ATFS [17], and therefore, the
TMCF Volume-Optimized design can take advantage of a first-order geometry that the TMCA cannot adequately correct, and therefore achieves a lower RMS WFE. The optimal first-order layout when using freeform surfaces is different than the optimal layout using rotationally symmetric surfaces, as seen by comparing the two layouts in Figure 4.4a and Figure 4.4b with the layout in Figure 4.4c.

![Figure 4.4. Layouts of each design type corresponding to a volume of 72.5 L, the smallest diffraction-limited volume of the TMCF Frozen-Geometry: (a) The TMCA, (b) TMCF Frozen-Geometry, and (c) the TMCF Volume-Optimized designs. Note that the apparent overlap of the surfaces in the layouts is due to the extension of the surfaces in the drawing program. The clear apertures and the rays have no conflicts in the designs.](image)

The level and type of aberration correction can be seen by examining the full-field displays (FFD) of the Zernike aberrations for each of these designs. An FFD represents the magnitude and, where appropriate, orientation of a given aberration across the full field-of-view using a symbol. In general, comparing the TMCA and TMCF Frozen-Geometry designs, we see that the TMCF Frozen-Geometry design is better able to correct the plane-symmetric aberrations, or when correction is not possible (or not optimal), to achieve a better balance of the field-dependence. For example, Figure 4.5 shows the Zernike defocus FFD. The TMCA design has some higher-order field curvature, while the TMCF Frozen-
Geometry design is able to substantially correct this aberration through balancing with a combination of $Z_4$, $Z_8$, $Z_9$, $Z_{12}$, and $Z_{15}$ Zernike surface coefficients. Figure 4.6 shows that the astigmatism is substantially reduced in both TMCF designs, with two astigmatism nodes brought into the FOV. The same is true for coma in Figure 4.7 and spherical in Figure 4.8. Furthermore, the TMCF Volume-Optimized design is able to achieve a better balance of these aberrations by changing the first-order layout. Additionally, the TMCF designs are able to directly correct the field-constant elliptical coma present in the TMCA design as seen in Figure 4.9.

![Figure 4.5. Zernike defocus (Z4) FFD for the (a) TMCA design, (b) TMCF Frozen-Geometry design, and (c) TMCF Volume-Optimized design.](image)

![Figure 4.6. Zernike Astigmatism (Z5/Z6) FFD for the (a) TMCA design, (b) TMCF Frozen-Geometry design, and (c) TMCF Volume-Optimized design.](image)
Not only do the freeform surfaces allow for better correction of the rotationally non-symmetric aberrations, but they also allow for better correction of rotationally symmetric aberrations because of their beam-centered coordinate reference. Since the Zernike surface departure is centered on the surface, the rotationally symmetric Zernike terms (Z4, Z9, Z16, Z25) can contribute directly to the correction of rotationally symmetric aberration terms,
in contrast to the off-axis asphere sections, where the rotationally symmetric contributions are not separable from the plane-symmetric contributions. Similarly, the plane-symmetric terms can contribute directly to the correction of plane-symmetric aberrations as determined by the ATFS [17]. For example, Figure 4.8a shows the Zernike spherical aberration for which there is a substantial uncorrected field-constant term, but this can be removed directly using a Z9 surface shape at the stop surface (or any other surface, for that matter [17,30]).

As seen in Table 4.3, the surface shapes in the TMCF designs depart further from a base sphere in general. Except for M1 of the TMCA, the freeform surfaces depart more than the off-axis aspheres. This extra departure is directly related to the centered nature of the Zernike surfaces. When using off-axis aspheres, the surface coefficients are not orthogonal and are furthermore coupled together in terms of their aberration correction abilities. However, the centered Zernike surface terms are able to independently add surface departure and aberration correction, and thus achieve better correction, while adding more departure. This is the inherent tradeoff in achieving better correction through freeform surfaces, though recent work has shown that the effect can be reduced by constraining the surface coefficient magnitudes without sacrificing much correction ability [72].
Table 4.3. Departure from Base Sphere for each design type in the iso-volume comparison

<table>
<thead>
<tr>
<th></th>
<th>TMCA</th>
<th>TMCF Frozen-Geometry</th>
<th>TMCF Volume-Optimized</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>370</td>
<td>283</td>
<td>251</td>
</tr>
<tr>
<td>M2</td>
<td>20</td>
<td>172</td>
<td>239</td>
</tr>
<tr>
<td>M3</td>
<td>12</td>
<td>264</td>
<td>175</td>
</tr>
</tbody>
</table>

4.5.2 Iso-performance comparison

Another illustrative comparison is between TMC designs of the same performance. The layouts of the three designs closest to 0.07 waves RMS WFE for each design type are shown in Figure 4.10. The volumes of each bounding box are 96.9 L, 72.5 L, and 59.1 L in Figures 4.10(a), 4.10(b), and 4.10(c), respectively. The bounding boxes are also shown overlaid in Figure 4.10(d).

![Figure 4.10. Layouts of the smallest volume diffraction-limited designs of each TMC type with their bounding boxes: (a) The TMCA is shown with a blue bounding box, (b) the TMCF Frozen-Geometry is shown with a red bounding box, and (c) the TMCF Volume-Optimized is shown with a green bounding box. (d) The bounding boxes are shown next to each other for perspective.](image)

The FFDs for each design type are shown in Figure 4.11-4.13. Though these designs share similar overall RMS WFE performance, they achieve a different balance of aberrations, similar to the iso-volume comparison in Section 4.5.1. Notably, the coma and higher order field-curvature are substantially reduced in both TMCF designs, while the plane-symmetric astigmatism terms actually increase for the TMCF designs. Evidently, the
balance of aberrations in the TMCF designs allows the first-order layout to shift to a lower volume configuration.

Figure 4.11. Zernike defocus (Z4) FFD for the: (a) TMCA design, (b) TMCF Frozen-Geometry design, and (c) TMCF Volume-Optimized design. Blue indicates a positive value, red indicates a negative value.

Figure 4.12. Zernike Astigmatism (Z5/Z6) FFD for the: (a) TMCA design, (b) TMCF Frozen-Geometry design, and (c) TMCF Volume-Optimized design.

Figure 4.13. Zernike coma (Z7/Z8) FFD for the: (a) TMCA design, (b) TMCF Frozen-Geometry design, and (c) TMCF Volume-Optimized design.

The departure from base sphere for each design type for the iso-performance case is shown in Table 4.4. We are showing the surface departure as a way of comparing the aspheric nature of each surface. For surfaces with slowly varying shapes like the low-order
Zernike polynomials used in this article, the sag and slope are closely correlated, so we have shown the maximum sag departure from base sphere. The TMCA design has substantially lower departure than either TMCF designs due largely to its larger volume and therefore slower surfaces. The surface departures of the primaries in all geometries contribute most of the departure, but the departure in both TMCF geometries are more balanced, and thus the aberration correction happening at each surface is more balanced as well. This trade-off in volume vs departure is not inherent to freeform, but it is accentuated by it. Faster optical surfaces are required for smaller volumes, and tilting those faster optical surfaces creates more off-axis aberrations, resulting in more required departure to correct those aberrations. However, freeform surfaces have more degrees of freedom, allowing for more modes of departure, and therefore can achieve the larger departures required to maintain performance at a given volume compared to off-axis sections of rotationally symmetric parent surfaces. The full Zernike coefficients are listed in Tables 4.5-4.7 in the Chapter 4 Appendix for reference.

<table>
<thead>
<tr>
<th>Mirror</th>
<th>TMCA</th>
<th>TMCF Frozen-Geometry</th>
<th>TMCF Volume-Optimized</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>208</td>
<td>310</td>
<td>488</td>
</tr>
<tr>
<td>M2</td>
<td>14</td>
<td>177</td>
<td>205</td>
</tr>
<tr>
<td>M3</td>
<td>20</td>
<td>204</td>
<td>169</td>
</tr>
</tbody>
</table>

### 4.6 Conclusion to Section 4

The ability of freeform surfaces to expand the design space for plane-symmetric unobscured optical systems allows better overall correction of plane-symmetric aberrations for a given volume resulting in greater performance for a given volume. This has been
demonstrated by progressively reducing the volume of a TMC design that uses rotationally symmetric surfaces and converting the layout to an equivalent field-centered freeform design that is shown to better correct the aberrations. Additionally, freeform surfaces allow for more compact first-order layouts, which are not otherwise correctable using rotationally symmetric surfaces. This was demonstrated by allowing the first-order layout of the converted designs to vary and achieving not only better correction by up to 70%, but smaller volume as well by up to 39%. Finally, a reduced volume comes at the expense of more freeform departures from the base sphere by up to an order of magnitude.
### 4.7 Chapter 4 Appendix

Table 4.5. The FRINGE Zernike coefficients for the primary mirror representing departure from base sphere for each diffraction-limited design.

<table>
<thead>
<tr>
<th>TMCA 96.9 L</th>
<th>TMCF Frozen-Geometry 72.5 L</th>
<th>TMCF Volume-Optimized 59.1 L</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum Aperture Semi-Diameter (mm)</td>
<td>176.3330</td>
<td>176.3297</td>
</tr>
<tr>
<td>Normalization Aperture Semi-Diameter (mm)</td>
<td>170.7363</td>
<td>176.3297</td>
</tr>
<tr>
<td>FRINGE Zernike Coefficient, Primary Mirror (mm)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z1</td>
<td>0.0002458149</td>
<td>-0.0499511345</td>
</tr>
<tr>
<td>Z2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z3</td>
<td>0.1165349914</td>
<td>0.2743568100</td>
</tr>
<tr>
<td>Z4</td>
<td>0.0000693106</td>
<td>-0.0745891417</td>
</tr>
<tr>
<td>Z5</td>
<td>-0.1353639608</td>
<td>-0.1303619649</td>
</tr>
<tr>
<td>Z6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z7</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z8</td>
<td>0.0580866222</td>
<td>0.1368588872</td>
</tr>
<tr>
<td>Z9</td>
<td>0.0042482223</td>
<td>-0.0244205099</td>
</tr>
<tr>
<td>Z10</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z11</td>
<td>0.0021226646</td>
<td>-0.0111876327</td>
</tr>
<tr>
<td>Z12</td>
<td>0.0008326605</td>
<td>0.0032969007</td>
</tr>
<tr>
<td>Z13</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z14</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z15</td>
<td>-0.0001083608</td>
<td>-0.0002331794</td>
</tr>
<tr>
<td>Z16</td>
<td>0.0000027475</td>
<td>0.002118401</td>
</tr>
<tr>
<td>Z17</td>
<td>0.0000613417</td>
<td>-0.000150526</td>
</tr>
<tr>
<td>Z18</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z19</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z20</td>
<td>-0.0000238379</td>
<td>-0.0001049851</td>
</tr>
<tr>
<td>Z21</td>
<td>-0.0000058420</td>
<td>-0.000020773</td>
</tr>
<tr>
<td>Z22</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z23</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z24</td>
<td>0.0000014837</td>
<td>-0.0000151256</td>
</tr>
<tr>
<td>Z25</td>
<td>-0.0000009877</td>
<td>-0.0000056571</td>
</tr>
<tr>
<td>Z26</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z27</td>
<td>-0.0000051085</td>
<td>0</td>
</tr>
<tr>
<td>Z28</td>
<td>-0.0000011038</td>
<td>0</td>
</tr>
<tr>
<td>Z29</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z30</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z31</td>
<td>-0.0000002160</td>
<td>0</td>
</tr>
<tr>
<td>Z32</td>
<td>0.0000004731</td>
<td>0</td>
</tr>
<tr>
<td>Z33</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z34</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z35</td>
<td>-0.0000007115</td>
<td>0</td>
</tr>
<tr>
<td>Z36</td>
<td>-0.0000008351</td>
<td>0</td>
</tr>
<tr>
<td>Z37</td>
<td>-0.0000003518</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 4.6. The FRINGE Zernike coefficients for the secondary mirror representing departure from base sphere for each diffraction-limited design.

<table>
<thead>
<tr>
<th></th>
<th>TMCA 96.9 L</th>
<th>TMCF Frozen-Geometry 72.5 L</th>
<th>TMCF Volume-Optimized 59.1 L</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum Aperture Semi-Diameter (mm)</td>
<td>58.8430</td>
<td>58.1558</td>
<td>51.8783</td>
</tr>
<tr>
<td>Normalization Aperture Semi-Diameter (mm)</td>
<td>59.8540</td>
<td>58.1558</td>
<td>51.8783</td>
</tr>
<tr>
<td>FRINGE Zernike Coefficient, Secondary Mirror (mm)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z1</td>
<td>-0.0000370732</td>
<td>0.0040222707</td>
<td>-0.0432379214</td>
</tr>
<tr>
<td>Z2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z3</td>
<td>-0.0108838775</td>
<td>0.0309447052</td>
<td>0.0519336946</td>
</tr>
<tr>
<td>Z4</td>
<td>-0.0000594983</td>
<td>0.0060388280</td>
<td>-0.0641484832</td>
</tr>
<tr>
<td>Z5</td>
<td>0.0075685216</td>
<td>0.1637450199</td>
<td>0.0459991873</td>
</tr>
<tr>
<td>Z6</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z7</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z8</td>
<td>-0.0056694096</td>
<td>0.0152122037</td>
<td>0.0258454803</td>
</tr>
<tr>
<td>Z9</td>
<td>-0.0009023299</td>
<td>0.020221364</td>
<td>-0.0204517791</td>
</tr>
<tr>
<td>Z10</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z11</td>
<td>0.0004316819</td>
<td>-0.0129031757</td>
<td>-0.0079094718</td>
</tr>
<tr>
<td>Z12</td>
<td>0.0003507853</td>
<td>0.0014483055</td>
<td>0.0008160642</td>
</tr>
<tr>
<td>Z13</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z14</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z15</td>
<td>-0.0001534008</td>
<td>-0.001660436</td>
<td>-0.0000823412</td>
</tr>
<tr>
<td>Z16</td>
<td>-0.000014538</td>
<td>0.0000070524</td>
<td>0.000486702</td>
</tr>
<tr>
<td>Z17</td>
<td>0.0000094924</td>
<td>-0.0001867808</td>
<td>-0.0001358152</td>
</tr>
<tr>
<td>Z18</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z19</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z20</td>
<td>-0.00000069325</td>
<td>-0.0000450512</td>
<td>-0.0000314659</td>
</tr>
<tr>
<td>Z21</td>
<td>-0.0000014708</td>
<td>-0.0000007223</td>
<td>-0.0000021700</td>
</tr>
<tr>
<td>Z22</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z23</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z24</td>
<td>-0.0000007616</td>
<td>0.0000055418</td>
<td>-0.0000010724</td>
</tr>
<tr>
<td>Z25</td>
<td>-0.0000000722</td>
<td>0.0000014733</td>
<td>-0.0000101124</td>
</tr>
<tr>
<td>Z26</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z27</td>
<td>-0.0000019708</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z28</td>
<td>-0.0000020352</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z29</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z30</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z31</td>
<td>0.0000008731</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z32</td>
<td>0.0000001185</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z33</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z34</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z35</td>
<td>-0.0000003232</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z36</td>
<td>0.0000001155</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z37</td>
<td>0.0000001151</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 4.7. The FRINGE Zernike coefficients for the tertiary mirror representing departure from base sphere for each diffraction-limited design.

<table>
<thead>
<tr>
<th></th>
<th>TMCA 96.9 L</th>
<th>TMCF Frozen-Geometry 72.5 L</th>
<th>TMCF Volume-Optimized 59.1 L</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum Aperture Semi-Diameter (mm)</td>
<td>130.8382</td>
<td>112.8335</td>
<td>95.8424</td>
</tr>
<tr>
<td>Normalization Aperture Semi-Diameter (mm)</td>
<td>127.0619</td>
<td>112.8335</td>
<td>95.8424</td>
</tr>
<tr>
<td>FRINGE Zernike Coefficient, Tertiary Mirror (mm)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z1</td>
<td>-0.0000295027</td>
<td>-0.0238951186</td>
<td>-0.0038304336</td>
</tr>
<tr>
<td>Z2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z3</td>
<td>0.0184948453</td>
<td>0.0296065412</td>
<td>0.0334096363</td>
</tr>
<tr>
<td>Z4</td>
<td>0.0000075512</td>
<td>-0.0359900827</td>
<td>-0.0057675680</td>
</tr>
<tr>
<td>Z5</td>
<td>0.0052708109</td>
<td>0.1735489688</td>
<td>0.1215498815</td>
</tr>
<tr>
<td>Z6</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z7</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z8</td>
<td>0.0093525473</td>
<td>0.0148120812</td>
<td>0.0167007000</td>
</tr>
<tr>
<td>Z9</td>
<td>-0.0030420183</td>
<td>-0.0121925764</td>
<td>-0.0019534687</td>
</tr>
<tr>
<td>Z10</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z11</td>
<td>0.0000009640</td>
<td>-0.0176368745</td>
<td>-0.0117847907</td>
</tr>
<tr>
<td>Z12</td>
<td>0.0000197513</td>
<td>0.0014581633</td>
<td>0.0007988727</td>
</tr>
<tr>
<td>Z13</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z14</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z15</td>
<td>0.0000638552</td>
<td>0.0000104340</td>
<td>-0.0000074654</td>
</tr>
<tr>
<td>Z16</td>
<td>-0.0000270045</td>
<td>-0.0000971190</td>
<td>-0.0000176261</td>
</tr>
<tr>
<td>Z17</td>
<td>0.0000038198</td>
<td>-0.0006640551</td>
<td>-0.0004828835</td>
</tr>
<tr>
<td>Z18</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z19</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z20</td>
<td>0.0000039500</td>
<td>-0.0001134695</td>
<td>-0.0000401195</td>
</tr>
<tr>
<td>Z21</td>
<td>-0.0000051920</td>
<td>0.0000000153</td>
<td>-0.0000066702</td>
</tr>
<tr>
<td>Z22</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z23</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z24</td>
<td>-0.0000049411</td>
<td>0.0000034202</td>
<td>-0.0000035400</td>
</tr>
<tr>
<td>Z25</td>
<td>0.0000008578</td>
<td>0.0000004933</td>
<td>-0.0000012918</td>
</tr>
<tr>
<td>Z26</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z27</td>
<td>-0.0000003919</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z28</td>
<td>0.00000026877</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z29</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z30</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z31</td>
<td>0.00000026547</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z32</td>
<td>-0.0000027713</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z33</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z34</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z35</td>
<td>-0.0000016328</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z36</td>
<td>-0.0000014470</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z37</td>
<td>-0.0000013066</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
5 Conclusion and Future Work

In this work, we have developed methods for the design and alignment of two classes of unobscured reflective optical systems. We have explored the theoretical design of two design forms of a laser relay; one using off-axis conic sections and one using tilted spherical components. In designing the off-axis conic relay, we showed the relationship between a Cartesian reflector’s Gaussian curvature and the effective radius of curvature, allowing the use of simple first-order equations to determine optical properties along the off-axis chief-ray. For the tilted-component relay, we showed the theoretical basis for the correction of field-constant astigmatism and field-constant coma using nodal aberration theory (NAT). We also extended the NAT astigmatism terms to $8^{th}$ order to ascertain the higher-order field dependence of the Zernike astigmatism aberrations, and showed how to use this field-dependence to aid in the design of a freeform telescope using the aberration theory of freeform surfaces (ATFS). This method allows one to estimate the surface coefficients required to simultaneously correct certain low-order aberrations. Finally, we showed the quantitative effect of freeform surfaces on the volume and performance of a three-mirror telescope design type called the three-mirror compact (TMC).

Looking towards future research paths, there are myriad routes to expand upon this research. First, the quantitative NAT-based ratio method for freeform design could be expanded to more Zernike surface shapes that are already included in ATFS. Such expansion would necessarily require designs with more surfaces to fully correct the aberrations predicted by ATFS. However, one could also take a more subtle approach, and determine the optimal aberration balance using a given number of surfaces. It would be
interesting to compare such an approach to a numerical optimization algorithm to see if there is some distinct advantage.

Additionally, regarding the design of the achromatic image relay (AIR), a useful next step to advance the goals of the MTW-OPAL project would be to determine the scalability of the all-spherical AIR design. It stands to reason that scaling up such a design to handle larger beams would necessitate non-spherical components once the beams become large enough. It would be useful to understand what that point might be. One could study this using a numerical optimization design study, but it may be worthwhile to fully characterize the NAT aberrations for this type of four-mirror design because this would make it easier to explore the design space for different beam sizes.
References

42. R. A. Buchroeder, "Design Examples of Tilted-Component Telescopes (TCT’s) (A Class of Unobscured Reflectors)," in Optical Sciences Technical Report, No. 68 (Optical Sciences Center, University of Arizona, 1971).
47. H. Coddington, A Treatise on the Reflexion and Refraction of Light (Cambridge, 1829).
57. K. P. Thompson and J. P. Rolland, "A page from “the drawer”: how Roland Shack opened the door to the aberration theory of freeform optics," in Fifty Years of Optical