MHD Equilibria with Poloidal and Toroidal Flow

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Outline

- Review of theory and numerics (with the code FLOW\(^1\)) of tokamak equilibria with flow

- Equilibria with toroidal flow, effects of flow and anisotropy (application to NSTX)

- Equilibria with poloidal flow:
  - Equilibria with transonic poloidal flow
  - Equilibria with super-Alfvénic poloidal flow
    - inward-shifted equilibria
    - quasi-omnigenous equilibria

FLOW web site: http://www.me.rochester.edu/~guazzott/FLOW_manual.htm
MHD equilibrium equations with flow

- Continuity:
  \[ \nabla \cdot (\rho \vec{v}) = 0 \]

- Momentum:
  \[ \rho \vec{v} \cdot \nabla \vec{v} = \vec{J} \times \vec{B} - \nabla \cdot \vec{P} \]
  \[ \vec{P} \equiv p_\perp \vec{I} + \Delta \vec{B} \vec{B} \quad \Delta \equiv (p_\parallel - p_\perp)/B^2 \]

- Maxwell equations (and Ohm’s law):
  \[ \nabla \times (\vec{v} \times \vec{B}) = 0 \quad \nabla \cdot \vec{B} = 0 \]
The MHD equations are reduced to a “Bernoulli” and a “Grad–Shafranov” equation

• Plasma flow

\[ \mathbf{\bar{v}} = M_{A\theta} \mathbf{\bar{v}}_A + R \Omega(\Psi) \mathbf{\hat{e}_\phi} \]

\[ M_{A\theta} = \frac{V_\theta}{V_{A\theta}} = \frac{\Phi(\Psi)}{\sqrt{\rho}} \]

• “Bernoulli” equation

\[ \frac{1}{2} \frac{(M_{A\theta}B)^2}{\rho} - \frac{1}{2} \left[ R \Omega(\Psi) \right]^2 + W = H(\Psi) \]

• “GS” equation

\[ \nabla \cdot \left[ \left( 1 - M_{A\theta}^2 - \Delta \right) \left( \nabla \Psi \right) \right] = \]

\[ - \frac{\partial p}{\partial \Psi} - \frac{B_\phi}{R} \frac{d F(\Psi)}{d \Psi} - \mathbf{v} \cdot \nabla \Phi(\Psi) \frac{d \Omega(\Psi)}{d \Psi} - R \rho v_\phi \frac{d \Phi(\Psi)}{d \Psi} - \rho \frac{d H(\Psi)}{d \Psi} + \rho \frac{\partial W}{\partial \Psi} \]

The input of the code FLOW uses quasi-physical free functions

- Each physical quantity reduces to the corresponding free function in the cylindrical limit.
- The input functions can be supplied as analytical expressions or numerical tables.

\[
\begin{align*}
D(\Psi) & \rightarrow \text{Quasi-density} \\
P_{||}(\Psi) & \rightarrow \text{Quasi-parallel pressure} \\
P_{\perp}(\Psi) & \rightarrow \text{Quasi-perpendicular pressure} \\
B_0(\Psi) & \rightarrow \text{Quasi-toroidal magnetic field} \\
M_{\theta}(\Psi) & \rightarrow \text{Quasi-poloidal sonic Mach number} \\
M_{\phi}(\Psi) & \rightarrow \text{Quasi-toroidal sonic Mach number}
\end{align*}
\]

Equilibria with Purely Toroidal Flow. Applications to NSTX

\[ \mathbf{v}_\phi = \Omega(\Psi) R \]

\[ \mathbf{v}_\theta = 0 \]

The centrifugal force causes an outward shift of the plasma.

Effect of increasing rotation with constant plasma total mass for NSTX-like equilibria.
The parallel anisotrophy \((p_\parallel > p_\perp)\) causes an inward shift.

\[
\Theta = \frac{P_\parallel(\Psi) - P_\perp(\Psi)}{P_\perp(\Psi)}
\]

(Toroidal Mach number)

\(M^c_\Phi = 0.5\)

Total energy is conserved.

NSTX-like parameters
Equilibria with Poloidal Flow
Viscosity is reduced in supersonic flows and omnigenous B-field

- Poloidal flows in tokamaks are damped.
- Poloidal viscosity $\nu_\theta \sim 1/M_p^2$
  
  \[
  M_p = \frac{v_\theta}{C_{s\theta}}
  \]

  \[
  C_{s\theta} = C_s \frac{B_\theta}{B}
  \]

- Equilibria with supersonic poloidal flow have reduced viscosity.
- Omnigenous equilibria have low neoclassical viscosity.
- Specific applications shown for the UCLA Electric Tokamak;

  Results have general applicability.

Equilibria with transonic poloidal flow:
flow profile ranging from subsonic to supersonic

\[ v_\theta \sim C_{s\theta} = C_s \varepsilon/q \]
Transonic solution of Bernoulli equation is discontinuous and imposes constraints on the free functions.

- \( M_\theta(\Psi) \) cannot be chosen arbitrarily.
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- \( M_\theta(\Psi) \) cannot be chosen arbitrarily.
Numerical transonic equilibria exhibit an edge pedestal structure in the pressure profile

- Low-\( \beta \)
  ET equilibrium
- Results are general
A bifurcated equilibrium exists for a critical poloidal velocity.

A bifurcated equilibrium exists when the free function $M_\theta(\Psi)$ reaches the critical value.

![Diagram showing bifurcated equilibrium](image)
A bifurcated equilibrium exists when the free function $M_\theta(\Psi)$ reaches the critical value.
A bifurcated equilibrium exists for a critical poloidal velocity

A bifurcated equilibrium exists when the free function $M_\theta(\Psi)$ reaches the critical value.

\[ \theta = 0 \]

\[ V_\theta = (\text{km/s}) \]

\[ M_\theta = (\Psi) \]

\[ M_\theta^{\text{MAX}} \]
Initial value 2-D MHD simulations of transonic flow show the generation of discontinuities

- Poloidal sound speed (red) and poloidal velocity (blue) evolve to a discontinuous state.

![Diagram showing poloidal sound speed and velocity profiles at different stages of the simulation.](image-url)
Equilibria with Super-Alfvénic Poloidal Flow

\[ v_\theta \geq V_{A\theta} \]
Equilibria with super-Alfvénic poloidal flow (with respect to the poloidal Alfvén speed)

Simple model

- Inverted Shafranov shift
- Poloidal flow $\gg$ toroidal flow

Quasi-omnigenous equilibria

- $|B| = |B| (\Psi) + O(\varepsilon^2)$
- Adjust flow to satisfy the above condition
- Small poloidal viscosity

Fully omnigenous equilibria\(^1\)

- $|B| = |B| (\Psi)$
- Negligible poloidal viscosity
- Not tokamak relevant

An analytic model gives inverted Shafranov shift for Super-Alfvénic equilibria

Assumptions:
\[ \varepsilon \ll 1 \quad \beta \sim \varepsilon \]
\[ M_{A\theta} \approx \text{const.} \quad \nu_\varphi \ll \nu_\theta \]

For a fixed \( \beta \) the Shafranov shift is computed as a function of \( M_{A\theta} \).

For inward shifted equilibria the pressure forces are balanced by the centrifugal forces

- Pressure forces cause outward Shafranov shift.
- Poloidal flow produces additional force.
- If the Shafranov shift is outward, pressure and centrifugal forces are aligned.
- If the Shafranov shift is inward, pressure and centrifugal forces are opposite.
FLOW confirms the existence of inward shifted equilibria for $M_{A\theta} > 1$

Numerical example:
- $M_{\phi} (\Psi) = 0$
- Flat density
- Peaked pressure
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![Diagram showing magnetic axis and R/R₀ vs. z/a with two graphs: one for $M_{A\theta}$ vs. R (m) and the other for R/R₀ vs. z/a.](image)
Quasi-omnigenous \([|B| = |B| (\Psi) + O(\varepsilon^2)]\) equilibria with fast poloidal flow

- The solution uses an \(\varepsilon\) expansion, assuming the ordering:

\[
B_\theta \sim \varepsilon B_\varphi \quad \left( M_{A\theta}^2 - 1 \right) \sim 1 \quad \beta \sim \varepsilon
\]

\[
|B|^2 = B_\varphi^2 + B_\theta^2 \approx B_\varphi^2 + O(\varepsilon^2)
\]

- \(B_\varphi\) is imposed to be a function of \(\Psi\) only up to \(O(\varepsilon^2)\) corrections.
Fast poloidal flows are used to make the magnetic field quasi-omnigenous.

In the static case

\[ B_\varphi = \frac{F(\Psi)}{R} \quad \rightarrow \quad B_\varphi = B_\varphi(\Psi) + O(\varepsilon). \]

With flow:

\[
B_\varphi = \frac{1}{1 - M_{A\theta}^2} \left[ \frac{F(\Psi)}{R} + R M_{A\theta} \sqrt{\rho \Omega(\Psi)} \right] \quad \rightarrow \quad \text{adjust flow to impose}
\]

\[ B_\varphi = B_\varphi(\Psi) + O(\varepsilon^2) \]

Graphs showing plots of \( M_\theta(\Psi) \), \( M_\varphi(\Psi) \), and \( B_0(\Psi) \) against \( \Psi_a \).
FLOW is used to compute the quasi-omnigenous equilibria of arbitrary shape
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Magnetic field relative variations are computed for a quasi-omnigenous and a static equilibrium.
Tokamak equilibria with flow are very different from static equilibria

- Equilibria with macroscopic flows have been studied analytically. The numerical results of the code FLOW confirm the results of theory.

- Equilibria with poloidal flow in the range of the poloidal sound speed $C_s B_\theta / B$ develop a pedestal structure due to the transition from subsonic to supersonic regime.

- Equilibria with poloidal flow in the super-Alfvénic regime show inverted Shafranov shift. The existence of a new class of quasi-omnigenous equilibria in such regime has been discussed.