Axisymmetric MHD Equilibria with Arbitrary Flow and Applications to NSTX

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Outline

• The code FLOW:
  – the system of equations
  – the numerical solution
• NSTX-like equilibria with toroidal flow
• NSTX-like equilibria with poloidal flow
• Conclusions
The relevant equations

- Continuity:

\[ \nabla \cdot (\rho \vec{v}) = 0 \]

- Momentum:

\[ \rho \vec{v} \cdot \nabla \vec{v} = \vec{J} \times \vec{B} - \nabla \cdot \vec{P} \]

\[ \vec{P} = p_\perp \vec{I} + \Delta B \vec{B} \]

\[ \Delta \equiv (p_\parallel - p_\perp)/B^2 \]

- Maxwell equations:

\[ \nabla \times (\vec{v} \times \vec{B}) = 0 \]

\[ \nabla \cdot \vec{B} = 0 \]
The previous system of equations can be reduced to a “Bernoulli” and a “Grad–Shafranov” equation

- Faraday’s law yields the plasma flow:

  \[ \vec{v} = M_{A\theta} \vec{v}_A + R\Omega(\Psi)\hat{\phi} \quad \quad \quad M_{A\theta} = \Phi(\Psi)/\sqrt{\rho} \]

- The \( \phi \)-component of the momentum equation gives an equation for the toroidal component of the magnetic field:

  \[ B_\phi R = \frac{I(\Psi) - R^2 M_{A\theta} \sqrt{\rho} \Omega(\Psi)}{1 - M_{A\theta}^2 - \Delta} \]

- The B-component of the momentum equation reduces to a “Bernoulli-like” equation for the total energy along the field lines:

  \[ \frac{1}{2} \left( \frac{M_{A\theta} B}{\rho} \right)^2 - \frac{1}{2} \left[ R\Omega(\Psi) \right]^2 + W = H(\Psi) \]
The modified Grad–Shafranov equation

- Finally, the $\nabla \Psi$-component of the momentum equation gives a “GS-like” equation:

$$ \nabla \cdot \left[ \left( 1 - M_{A\theta}^2 - \Delta \right) \left( \frac{\nabla \Psi}{R^2} \right) \right] $$

$$ = - \frac{\partial p_\parallel}{\partial \Psi} - \frac{B_\phi}{R} \frac{dI(\Psi)}{d\Psi} - \vec{v} \cdot \vec{B} \frac{d\Phi(\Psi)}{d\Psi} - R \rho v_\phi \frac{d\Omega(\Psi)}{d\Psi} - \rho \frac{dH(\Psi)}{d\Psi} + \rho \frac{dW}{d\Psi} $$

- $W (\rho, B, \Psi)$ is the enthalpy of the plasma, and its definition depends on the description of the plasma (MHD, CGL, or kinetic).

- $I(\Psi), \Phi(\Psi), \Omega(\Psi), H(\Psi), (\partial p_\parallel/\partial \Psi), (\partial W/\partial \Psi)$ are free functions of $\Psi$. 

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The code input requires a “user-friendly” set of free functions

- The input is a set of free functions representing quasi-physical variables.
- Functions can be supplied as analytical expressions or numerical tables.

\[ D(\Psi') \rightarrow \text{quasi-density} \]
\[ P_{||}(\Psi') \rightarrow \text{quasi-parallel pressure} \]
\[ P_{\perp}(\Psi') \rightarrow \text{quasi-perpendicular pressure} \]
\[ B_0(\Psi') \rightarrow \text{quasi-toroidal magnetic field} \]
\[ M_{\theta}(\Psi') \rightarrow \text{quasi-poloidal sonic Mach number} \]
\[ M_{\phi}(\Psi') \rightarrow \text{quasi-toroidal sonic Mach number} \]
The numerical algorithm: the multi-grid solver

- The Bernoulli equation is solved for $\rho$.
- The Grad-Shafranov equation is solved for $\Psi$ using a red-black algorithm.
- If the system is anisotropic, the equation for $B_\varphi$ is also solved.
- The procedure is repeated until convergence; then the solution is interpolated onto the next grid.
NSTX-like equilibria with toroidal flow

• The following set of free functions is used as input to compute anisotropic equilibria with toroidal flow:

\[
D(\Psi) = D^C \sqrt{\Psi} \quad \quad \quad P_{||}(\Psi) = P^C \Psi^2
\]

\[
B_0(\Psi) = \delta B_0 \Psi^3 + B_{\text{vacuum}} \quad \quad \quad P_{\perp}(\Psi) \leq P_{||}(\Psi)
\]

\[
M_\theta(\Psi) = 0 \quad \quad \quad M_\phi(\Psi) = M^C_\phi \sqrt{\Psi}
\]

\[
D^C = 3.4 \times 10^{19} \, (\text{m}^{-3}) \quad \quad \quad B^C_0 = 0.29 \, (\text{T}) \quad \quad \quad 0 \leq M^C_\phi \leq 2.5
\]

\[
\beta_T = 9\% \quad \quad \quad I = 0.9 \, (\text{MA})
\]

\[
R_0 = 0.86 \, (\text{m}) \quad \quad \quad a = 0.69 \, (\text{m}) \quad \quad \quad k = 1.9
\]

The centrifugal force causes an outward shift of the plasma.
The parallel anisotropy \((p_\parallel > p_\perp)\) causes an inward shift.
Flow and parallel anisotropy have opposite effect
User-supplied tables can be used as input

- Input functions from numerical data:
  \[ \Omega(\Psi) \]
  \[ P(\Psi) \]

\[ M_{\phi}^{\text{max}} \approx 0.4 \]

A strongly shaped, NSTX-like equilibrium
FLOW can also be applied to equilibria with poloidal flow

- The free function determining the poloidal flow is $M_{\theta}(\Psi)$ representing (approximately) the sonic poloidal Mach number (poloidal velocity/poloidal sound speed). Poloidal sound speed $= C_S B_{\theta}/B$.

- Radial discontinuities in the equilibrium profiles develop when the poloidal flow becomes transonic [$M_{\theta}(\Psi) \sim 1$].

- Since the poloidal sound speed is small at the plasma edge, transonic flows may develop near the edge.

- FLOW can describe MHD or CGL equilibria with poloidal flow.
NSTX-like equilibria with poloidal flow can exhibit a pedestal structure at the edge.
Conclusions

• The code FLOW can compute axysymmetric anisotropic equilibria with arbitrary flow.

• MHD, CGL, and kinetic closures are implemented. At the moment, poloidal flow is described only by the MHD and CGL closures.

• Fast-rotating anisotropic NSTX-like equilibria have been computed with FLOW.