Effect of Electric Fields on Electron Thermal Transport in Laser-Produced Plasmas

Introduction

Thermal transport plays an important role in direct-drive inertial confinement fusion. The Spitzer–Härm heat flux[1] $\mathbf{q}_{\mathrm{SH}} = -\kappa_{\mathrm{SH}}\nabla T$ has been conventionally used in the direct-drive inertial confinement fusion (ICF) hydrocodes. Here, κ_{SH} is the Spitzer heat conductivity and T is the electron temperature. In the regions of the steep temperature gradients where $\mathbf{q}_{\mathrm{SH}}$ exceeds a fraction f of the free-stream limit $q_{\mathrm{FS}} = nTv_T$, the Spitzer flux is replaced[2] by fq_{FS}, where n is the electron density, $v_T = \sqrt{T/m}$ is the electron thermal velocity, and $f =$ 0.05 – 0.1 is the flux limiter. It has been known for more than two decades[3–5] that, in addition to the terms proportional to the temperature gradients (thermal terms), the heat flux in laser-produced plasmas contains ponderomotive terms that are due to the gradients in the laser electric field. To our best knowledge, no systematic analysis has been performed to address the effect of such terms on the hydrodynamic flow in ICF plasmas. As shown later, the ratio of the ponderomotive terms to the thermal terms is proportional to $R = \alpha\left(v_E/v_T\right)^2\left(L_T/L_E\right)$, where $v_E = eE/m\omega_L$ is the electron quiver velocity, e is the electric charge, E is the amplitude of the electric field, m is the electron mass, ω_L is the laser frequency, L_T and L_E are the temperature and the electric field scale length, and α is a constant. The ratio of the electron quiver velocity to the thermal velocity is small for typical plasma parameters. Indeed, $\left(v_E/v_T\right)^2 \simeq 0.4\ I_{15}\lambda_{\mu\mathrm{m}}^2\big/T_{\mathrm{keV}}$, where I_{15} is the laser intensity in 10^{15} W/cm^2, $\lambda_{\mu\mathrm{m}}$ is the laser wavelength in microns, and T_{keV} is the electron temperature in keV. Using $I_{15} \sim 1$ and $T \sim 2$ keV, we obtain $\left(v_E/v_T\right)^2 \sim 0.02$ for $\lambda_{\mu\mathrm{m}} =$ 0.353 μm. The ratio R, however, can be of the order of unity due to a large ratio L_T/L_E. Indeed, as the laser reaches the turning point where the electron density equals $n_c \cos^2\theta$, the electric field decays toward the overdense portion of the shell as[6] $E \sim E_{\max}\exp\left(-2/3\zeta^{3/2}\right)$, where $n_c = m_e\omega_L^2\big/4\pi e^2$ is the critical density, θ is the laser incidence angle, $\zeta = \left(\omega_L L_n/c\right)^{2/3} z/L_n$, $L_n \sim L_T$ is the electron-density scale length, and z is the coordinate along the density gradient. Therefore, the electric-field scale length near the turning point becomes $L_E \sim L_T\Big/\left(\omega_L L_T/c\right)^{2/3}$. Substituting this estimate to the ratio R and using $L_T \sim 10\ \mu$m and $\left(v_E/v_T\right)^2 \sim 0.02$ gives $R \sim \alpha\left(v_E/v_T\right)^2\left(\omega_L L_T/c\right)^{2/3} \sim 0.6\ \alpha$. As will be shown later, the coefficient α is numerically large and proportional to the ion charge Z; this makes R larger than 1. This simple estimate shows that the ponderomotive terms become comparable to the thermal terms in the electron thermal flux near the turning point. In addition, the p-polarization of the electric field (polarization that has a field component directed along the density gradient) tunnels through the overdense portion of the shell and gives a resonance electric field at the critical surface.[6] The gradient of such a field is proportional to the ratio $\omega_L/\nu_{\mathrm{ei}}$, where ν_{ei} is the electron–ion collision frequency at the critical surface. Substituting typical direct-drive experiment parameters into an expression for the electron–ion collision frequency at the critical surface, $\nu_{\mathrm{ei}}\big/\omega_L \simeq 1.5 \times 10^{-3}\ Z\big/T_{\mathrm{keV}}^{3/2}$, shows a significant contribution of the ponderomotive terms to the heat flux near the critical surface.

In this article, the ponderomotive transport coefficients are derived. Such coefficients have been considered previously.[3–5,7–9] Reference 7 developed a method of solving the kinetic equation by separation of the electron distribution function on the high-frequency component due to the laser field and the low-frequency component of the time-averaged plasma response. Using such a method, the laser fields' contribution to the electron stress tensor was obtained. A similar method was used in Ref. 3, where the importance of the ponderomotive effects on the electron thermal conduction was emphasized. P. Mora and R. Pellat[4] and I. P. Shkarofsky[5] have evaluated the contributions of the laser fields into the heat and momentum fluxes. As was pointed out in Ref. 8, by not, however, consistently taking into account the contribution of the electron–electron collisions—the transport coefficients in their results contain wrong numerical factors. A consistent analysis was performed in Ref. 8, where results were obtained in the limit of large ion charge. Such a limit was relaxed in Ref. 9. The latter reference, however, contains numerous typographical errors, so the results will therefore be rederived in this article. The effect of ponderomotive terms on the hydrodynamic flow in direct-drive ICF experiments will be discussed in detail in a forthcoming publication.

Model

We consider a fully ionized plasma in a high-frequency electromagnetic field:

$$\varepsilon = \frac{1}{2}\left[\mathbf{E}(r,t)e^{-i\omega_L t} + \mathbf{E}(r,t)^* e^{i\omega_L t}\right], \tag{1}$$

$$\beta = \frac{1}{2}\left[\mathbf{B}(r,t)e^{-i\omega_L t} + \mathbf{B}(r,t)^* e^{i\omega_L t}\right], \tag{2}$$

where **E** and **B** are slowly varying (with respect to $e^{i\omega_L t}$) electric and magnetic fields and $\mathbf{E}^*$ and $\mathbf{B}^*$ are the complex conjugate (c.c.) of **E** and **B**. The electron distribution function f obeys the Boltzmann equation

$$\partial_t f + \mathbf{v}\partial_{\mathbf{r}} f + e\left(\mathbf{E}_0 + \varepsilon + \frac{\mathbf{v}\times\beta}{c}\right)\partial_{\mathbf{p}} f$$

$$= J_{\text{ee}}\left[f,f\right] + J_{\text{ei}}\left[f\right], \tag{3}$$

where $\mathbf{E}_0$ is the low-frequency electric field. Here,

$$J_{\text{ei}}\left[f\right] = 3\sqrt{\frac{\pi}{8}}\left(\frac{v_T}{v}\right)^3 \nu_{\text{ei}}\frac{\partial}{\partial v_k}\left[\left(v^2\delta_{kj} - v_k v_j\right)\frac{\partial f}{\partial v_j}\right] \tag{4}$$

is the ion–electron collision operator,

$$\nu_{\text{ei}} = \frac{4\sqrt{2\pi}e^4 n\bar{Z}\Lambda}{3m^2 v_T^3} \tag{5}$$

is the electron–ion collision frequency,

$$\bar{Z} = \sum_i e_i^2 n_i \Big/ \sum_i e_i n_i \tag{6}$$

is the average ion charge, n_i is the ion number density, n is the electron density, e_i is the ion charge, m is the electron mass, Λ is the Coulomb logarithm, $v_T = \sqrt{T/m}$, and T is the electron temperature. The sum in $\bar{Z}$ is taken over all ion species in the plasma. The electron–electron collision integral is taken in Landau form

$$J_{\text{ee}}\left[f,f\right] = \frac{2\pi e^4\Lambda}{m^2}$$

$$\times \frac{\partial}{\partial v_k}\int d\mathbf{v}'\frac{(\mathbf{v}-\mathbf{v}')^2\delta_{kj} - (v-v')_k(v-v')_j}{|\mathbf{v}-\mathbf{v}'|^3}$$

$$\times\left(\frac{\partial}{\partial v_j} - \frac{\partial}{\partial v_j'}\right)f(\mathbf{v})f(\mathbf{v}'). \tag{7}$$

Next, following Ref. 7, we separate the electron distribution function on the slowly varying part f_0 and the high-frequency component f_1:

$$f = f_0 + \frac{1}{2}\left(f_1 e^{-i\omega_L t} + f_1^* e^{i\omega_L t}\right). \tag{8}$$

Substituting Eqs. (1), (2), and (8) into Eq. (3) and collecting the terms with equal powers of $e^{i\omega_L t}$, we obtain

$$\partial_t f_1 - i\omega_L f_1 + \mathbf{v}\partial_{\mathbf{r}} f_1 + e\mathbf{E}_0\partial_{\mathbf{p}} f_1$$

$$+ e\mathbf{E}\partial_{\mathbf{p}} f_0 - \frac{ie}{\omega_L}\left[\mathbf{v}\times(\nabla\times\mathbf{E})\right]\partial_{\mathbf{p}} f_0$$

$$= J_{\text{ee}}\left[f_0,f_1\right] + J_{\text{ee}}\left[f_1,f_0\right] + J_{\text{ei}}\left[f_1\right], \tag{9}$$

$$\partial_t f_0 + \mathbf{v}\partial_{\mathbf{r}} f_0 + e\mathbf{E}_0\partial_{\mathbf{p}} f_0 - J_{\text{ei}}\left[f_0\right] - J_{\text{ee}}\left[f_0,f_0\right]$$

$$= \left\{-\frac{e}{4}\mathbf{E}\partial_{\mathbf{p}} f_1^* + \frac{ie}{4\omega_L}\left[\mathbf{v}\times(\nabla\times\mathbf{E})\right]\partial_{\mathbf{p}} f_1^*\right.$$

$$\left. + \frac{1}{4}J_{\text{ee}}\left[f_1,f_1^*\right] + \text{c.c.}\right\}. \tag{10}$$

Then, to relate f_1 with f_0, we assume that the laser frequency is high enough so f_1 can be expanded in series of

ω_L^{-1}: $f_1 = f_1^{(1)} + f_1^{(2)} + \cdots$, where $f_1^{(2)}/f_1^{(1)} \sim \nu_{\mathrm{ei}}/\omega_L \ll 1$. Substituting the latter expansion into Eq. (9) gives

$$f_1^{(1)} = -\frac{ie}{\omega_L}\mathbf{E}\partial_{\mathbf{p}} f_0, \tag{11}$$

$$f_1^{(2)} = -\frac{e}{\omega_L^2}\left\{\left(\partial_t + \mathbf{v}\partial_{\mathbf{r}}\right)\mathbf{E}\partial_{\mathbf{p}} f_0 - J_{\mathrm{ei}}\left[\mathbf{E}\partial_{\mathbf{p}} f_0\right]\right\}. \tag{12}$$

To eliminate f_1 from Eq. (10) for the low-frequency component of the distribution function, we substitute Eqs. (11) and (12) into Eq. (10). The result takes the form[8]

$$\begin{aligned}
&\partial_t f_0 + \mathbf{v}\partial_{\mathbf{r}} f_0 + e\mathbf{E}_0\partial_{\mathbf{p}} f_0 - J_{\mathrm{ei}}\left[f_0\right] - J_{\mathrm{ee}}\left[f_0, f_0\right] \\
&= \frac{e^2}{4\omega_L^2}\left\{\frac{\nabla|\mathbf{E}|^2}{m}\partial_{\mathbf{p}} f_0 + \frac{1}{2}\frac{\partial^2 f_0}{\partial p_i\,\partial p_j}\left(\partial_t + \mathbf{v}\partial_{\mathbf{r}}\right)\left(E_i E_j^* + \text{c.c.}\right)\right. \\
&\quad + \left(E_i E_j^* + \text{c.c.}\right)\left[\frac{1}{m}\frac{\partial^2 f_0}{\partial r_i\,\partial p_j} + \left(\partial_t + \mathbf{v}\partial_{\mathbf{r}}\right)\frac{\partial^2 f_0}{\partial p_i\,\partial p_j}\right. \\
&\quad \left.\left. - \partial_{p_i} J_{\mathrm{ei}}\left(\partial_{p_j} f_0\right) + J_{\mathrm{ee}}\left(\partial_{p_i} f_0, \partial_{p_j} f_0\right)\right]\right\}.
\end{aligned} \tag{13}$$

Equation (13) is solved assuming a small deviation of the electron distribution function f_0 from Maxwellian f_{M}:

$$f_0 = f_{\mathrm{M}}\left[1 + \psi\left(\mathbf{v}, \mathbf{p}, t\right)\right], \tag{14}$$

where $|\psi| \ll 1$. The kinetic equation for ψ is obtained by substituting the expansion (14) into Eq. (13) and replacing the time derivatives $\partial_t f_{\mathrm{M}}$ using the transport conservation equations. These equations, according to the standard procedure,[10] are obtained by multiplying the kinetic equations by $(\mathbf{v} - \mathbf{v}_0)^k$ with k = 0, 1, 2,... and integrating the latter in the velocity space. Here,

$$\mathbf{v}_0 = \frac{1}{\rho}\left(\int d\mathbf{v}\, m\mathbf{v} f_0 + \int d\mathbf{v}_i\, m_i \mathbf{v}_i f_i\right) \tag{15}$$

is the mass velocity, $\rho = nm + n_i m_i \simeq n_i m_i$ is the mass density, m_i is the ion mass, $\mathbf{v}_i$ is the ion velocity, and f_i is the ion distribution function. When k = 0, the described procedure yields the mass conservation equation; k = 1 and k = 2 give the momentum and energy conservation equations, respectively. Omitting lengthy algebraic manipulations we report the final result:[8]

$$\partial_t n + \nabla\left(n\mathbf{v}_0\right) + \nabla\left(n\mathbf{V}\right) = 0, \tag{16}$$

$$\begin{aligned}
\rho\partial_t v_{0k} + \rho\left(\mathbf{v}_0\partial_{\mathbf{r}}\right)v_{0k} &= -\partial_{r_k}\left(p_e + p_i\right) - \partial_{r_j}\sigma_{kj} + \rho_e E_{0k} \\
&\quad - \frac{e^2}{4m\omega_L^2}\left\{n_e\partial_{r_k}|E|^2 - \partial_{r_j}\left[n\left(E_k E_j^* + \text{c.c.}\right)\right]\right\},
\end{aligned} \tag{17}$$

$$\begin{aligned}
&\left(\partial_t + \mathbf{v}_0\partial_{\mathbf{r}}\right)\left(T + \frac{e^2|E|^2}{6m\omega_L^2}\right) - \frac{T}{n}\nabla\left(n\mathbf{V}\right) + \frac{2}{3n_e}\nabla\mathbf{q} + \frac{2p_e}{3n}\nabla\mathbf{v}_0 \\
&= \frac{m v_E^2 \nu_{\mathrm{ei}}}{3} + \frac{2e}{3}\mathbf{E}_0\mathbf{V} - \frac{2m}{m_i}\nu_{\mathrm{ei}}\left(T - T_i\right).
\end{aligned} \tag{18}$$

Here we use the standard definitions

$$n = \int d\mathbf{v} f_0, \quad n\mathbf{V} = \int d\mathbf{v}\left(\mathbf{v} - \mathbf{v}_0\right)f_0 = \mathbf{j}/e, \tag{19}$$

$$\mathbf{q} = \frac{m}{2}\int d\mathbf{v}\left(\mathbf{v} - \mathbf{v}_0\right)\left(\mathbf{v} - \mathbf{v}_0\right)^2 f_0, \tag{20}$$

$$\sigma_{kj} = m\int d\mathbf{v}\left(v - v_0\right)_k\left(v - v_0\right)_j f_0 - p_e\delta_{kj}, \tag{21}$$

where T_i is the ion temperature, n is the electron density, $\mathbf{j}$ is the current density, $\mathbf{q}$ is the heat flux, σ_{kj} is the stress tensor, p_e and p_i are the electron and ion pressures, $v_E = eE/m\omega_L$, and $\rho_e = en + e_i n_i$ is the charge density. To simplify the derivation of the transport coefficients, we assume $\mathbf{v}_0 = 0$ and neglect terms of the order of m/m_i. Next, the equation for the correction ψ to the Maxwellian distribution function is derived by substituting Eq. (14) into Eq. (13) and using the conservation equations (16)–(18). The resulting equation takes the form[8]

$$\partial_t\psi + \mathbf{v}\partial_{\mathbf{r}}\psi - \delta J_{\mathrm{ei}}[\psi] - \delta J_{\mathrm{ee}}[\psi]$$

$$= \left(\frac{3}{2} - x + \frac{3}{4}\sqrt{\frac{\pi}{x}}\right)\frac{\nu_{\mathrm{ei}}}{3}\frac{v_E^2}{v_T^2} + \left(x - \frac{3}{2}\right)\frac{2\nabla\mathbf{q}}{3nT} + \left(\frac{5}{2} - x\right)\frac{\nabla(n\mathbf{V})}{n}$$

$$+\mathbf{v}\left[\left(\frac{5}{2} - x\right)\nabla\ln T - \nabla\ln nT + \frac{e\mathbf{E}_0}{T} + \left(\frac{x}{3} - 1\right)\frac{\nabla v_E^2}{2v_T^2}\right]$$

$$+\frac{x}{10}v_k\frac{\partial_{r_j}\left(\mathbf{v}_E^2\right)_{jk}}{v_T^2} + \frac{3\nu_{\mathrm{ei}}}{8x^{3/2}}\frac{\left(\mathbf{v}^2\right)_{jk}\left(\mathbf{v}_E^2\right)_{jk}}{v_T^4}$$

$$\times\left[\frac{\sqrt{\pi}}{2}\left(1 + \frac{3}{2x}\right) + \frac{3}{\bar{Z}}\int_0^x dx' e^{-x'}\sqrt{x'}\left(1 - \frac{x'}{x}\right)\right]$$

$$+\frac{\left(\mathbf{v}^3\right)_{jkl}}{8v_T^4}\partial_{r_l}\left(\mathbf{v}_E^2\right)_{jk}, \tag{22}$$

where $x = v^2/\left(2v_T^2\right)$,

$$\delta J_{\mathrm{ei}}[\psi] = \frac{3\sqrt{\pi}}{8x^{3/2}}\nu_{\mathrm{ei}}\partial_{v_k}\left[\left(v^2\delta_{kj} - v_k v_j\right)\partial_{v_j}\psi\right], \tag{23}$$

$$\delta J_{\mathrm{ee}}[\psi] = \frac{3\sqrt{2\pi}\nu_{\mathrm{ei}}v_T^3}{4\bar{Z}n_e f_{\mathrm{M}}}\partial_{v_k}\left\{f_{\mathrm{M}}\int d\mathbf{v}' f_{\mathrm{M}}(v')\right.$$

$$\times\frac{(\mathbf{v}-\mathbf{v}')^2\delta_{kj} - (v-v')_k(v-v')_j}{|\mathbf{v}-\mathbf{v}'|^3}$$

$$\left.\times\left[\partial_{v_j}\psi(\mathbf{v}) - \partial_{v'_j}\psi(\mathbf{v}')\right]\right\}, \tag{24}$$

$$\left(\mathbf{v}^2\right)_{jk} \equiv v_j v_k - \frac{v^2}{3}\delta_{jk},$$

$$\left(\mathbf{v}^3\right)_{jkl} \equiv v_j v_k v_l - \frac{v^2}{5}\left(v_j\delta_{kl} + v_k\delta_{jl} + v_l\delta_{jk}\right), \tag{25}$$

and

$$\left(\mathbf{v}_E^2\right)_{jk} \equiv \frac{e^2}{\omega_L^2 m^2}\left(E_j E_k^* + E_j^* E_k - \frac{2}{3}\delta_{jk}|E|^2\right). \tag{26}$$

Next, we solve Eq. (22) assuming that the electron quiver velocity is much smaller than the electron thermal velocity, $v_E/v_T << 1$, and ordering $\nabla T/T \sim \nu_{\mathrm{ei}}v_E^2/v_T^3$. The function ψ is expanded as $\psi = \psi_1 + \psi_2 + \ldots$, where $\psi_2 << \psi_1$. The first approximation ψ_1 is obtained by keeping only the terms of the order of $v_T\nabla T/\left(T\nu_{\mathrm{ei}}\right)$. The second-order correction ψ_2 is derived by retaining the first derivative of the electric field and the second derivative of the electron temperature and density.

First-Order Approximation

Retaining the first spatial derivatives in temperature and density and also terms proportional to v_E^2/v_T^2, Eq. (22) yields

$$\delta J_{\mathrm{ei}}[\psi_1] + \delta J_{\mathrm{ee}}[\psi_1] = \left(x - \frac{3}{2} - \frac{3}{4}\sqrt{\frac{\pi}{x}}\right)\frac{\nu_{\mathrm{ei}}}{3}\frac{v_E^2}{v_T^2}$$

$$-\mathbf{v}\left[\left(\frac{5}{2} - x\right)\nabla\ln T - \nabla\ln nT + \frac{e\mathbf{E}_0}{T}\right] - \frac{3\nu_{\mathrm{ei}}}{8x^{3/2}}\frac{\left(\mathbf{v}^2\right)_{ij}\left(\mathbf{v}_E^2\right)_{ij}}{v_T^4}$$

$$\times\left[\frac{\sqrt{\pi}}{2}\left(1 + \frac{3}{2x}\right) + \frac{3}{\bar{Z}}\int_0^x dx' e^{-x'}\sqrt{x'}\left(1 - \frac{x'}{x}\right)\right]. \tag{27}$$

We look for a solution of Eq. (27) in the form

$$\psi_1 = \Phi_{11}(x)\frac{\left(\mathbf{v}^2\right)_{ij}\left(\mathbf{v}_E^2\right)_{ij}}{v_T^4} + \Phi_{12}\frac{v_E^2}{v_T^2}$$

$$+\Phi_{13}\frac{\mathbf{v}}{\nu_{\mathrm{ei}}}\nabla\ln T + \Phi_{14}\frac{\mathbf{v}}{\nu_{\mathrm{ei}}}\left[\nabla\ln nT - \frac{e\mathbf{E}_0}{T}\right]. \tag{28}$$

Using definitions (19)–(21), the current density, heat flux, and stress tensor in the first approximation become

$$\mathbf{j}^{(1)} = env_T\lambda_e\left[\alpha_{j1}^T\nabla\ln T + \alpha_{j2}^T\left(\nabla\ln nT - \frac{e\mathbf{E}_0}{T}\right)\right], \quad (29)$$

$$\mathbf{q}^{(1)} = nTv_T\lambda_e\left[\alpha_{q1}^T\nabla\ln T + \alpha_{q2}^T\left(\nabla\ln nT - \frac{e\mathbf{E}_0}{T}\right)\right], \quad (30)$$

$$\sigma_{kj}^{(1)} = \frac{4}{15}\alpha_\sigma^E mn\left(\mathbf{v}_E^2\right)_{kj}, \quad (31)$$

where $\lambda_e = v_T/\nu_{\rm ei}$ is the electron mean-free path. The numerical coefficients in Eqs. (29)–(31) have the forms

$$\alpha_{j1(j2)}^T = \frac{4}{3\sqrt{\pi}}\int_0^\infty dx x^{3/2}e^{-x}\Phi_{13(14)}(x), \quad (32)$$

$$\alpha_{q1(q2)}^T = \frac{4}{3\sqrt{\pi}}\int_0^\infty dx x^{5/2}e^{-x}\Phi_{13(14)}(x), \quad (33)$$

$$\alpha_\sigma^E = \frac{4}{\sqrt{\pi}}\int_0^\infty dx x^{5/2}e^{-x}\Phi_{11}(x). \quad (34)$$

Equations (29)–(31) show that the electric current and the heat flux in the first approximation are proportional to the gradients in temperature and pressure.[1,10,11] The stress tensor, on the other hand, depends on the laser electric field.[3,7] Even though the functions Φ_{11} and Φ_{22} do not enter into the first-order heat flux, they contribute to the heat flux in the second approximation. Thus, we need to find all four functions Φ_{11-14}. The general form of the solution ψ_1 [Eq. (28)] can be separated on the following three types of functions: type I depends only on the velocity modulus $\psi_1^{(\rm I)} = \Phi(x)$; type II is proportional to the velocity vector and velocity modulus $\psi_1^{(\rm II)} = A_j v_j\Phi(x)$; and type III depends on the velocity tensor and velocity modulus $\psi_1^{(\rm III)} = \left(\mathbf{v}^2\right)_{ij}\left(\mathbf{v}_E^2\right)_{ij}\Phi(x)$, where A_i is the vector proportional to the temperature, pressure gradients, or the electric field $\mathbf{E}_0$. According to such a classification, the governing equations for the functions of each type become

$$\text{Type I: } \delta J_{\rm ee}\left[\Phi(x)\right] = \phi(x), \quad (35)$$

$$\text{Type II: } A_j\left\{\Phi(x)\delta J_{\rm ei}\left[v_j\right] + \delta J_{\rm ee}\left[v_j\Phi(x)\right]\right\} = \phi(x)\mathbf{vA}, \quad (36)$$

$$\text{Type III: } \left(\mathbf{v}_E^2\right)_{ij}\left\{\Phi(x)\delta J_{\rm ei}\left[\left(\mathbf{v}^2\right)_{ij}\right] + \delta J_{\rm ee}\left[\left(\mathbf{v}^2\right)_{ij}\Phi(x)\right]\right\} = \phi(x)\left(\mathbf{v}^2\right)_{ij}\left(\mathbf{v}_E^2\right)_{ij}, \quad (37)$$

where $\phi(x)$ is defined by the right-hand side of Eq. (27). Since the ion–electron collision operator has a very simple form, it is straightforward to calculate $J_{\rm ei}\left[v_j\right]$ and $J_{\rm ei}\left[\left(\mathbf{v}^2\right)_{ij}\right]$ using Eq. (23):

$$\delta J_{\rm ei}\left[v_j\right] = -\frac{3\sqrt{\pi}}{4x^{3/2}}\nu_{\rm ei}v_j, \quad (38)$$

$$\delta J_{\rm ei}\left[\left(v^2\right)_{ij}\right] = -\frac{9\sqrt{\pi}}{4x^{3/2}}\nu_{\rm ei}\left(\mathbf{v}^2\right)_{ij}. \quad (39)$$

The electron–electron operator is more complicated, and the evaluation of $\delta J_{\rm ee}\left[\Phi(x)\right]$, $\delta J_{\rm ee}\left[v_j\Phi(x)\right]$, and $\delta J_{\rm ee}\left[\left(\mathbf{v}^2\right)_{ij}\Phi\right]$ requires lengthy algebra. Below is a detailed calculation of $\delta J_{\rm ee}\left[\Phi(x)\right]$. The integral part in the electron–electron collision operator can be rewritten as

$$\int d\mathbf{v}' f_{\rm M}(v')\frac{\left(\mathbf{v}-\mathbf{v}'\right)^2\delta_{kj} - \left(v-v'\right)_k\left(v-v'\right)_j}{\left|\mathbf{v}-\mathbf{v}'\right|^3} \times \left[\frac{v_j}{v_T^2}\Phi'(x) - \frac{v_j'}{v_T^2}\Phi'(x')\right] = \sum(x)v_k, \quad (40)$$

where function $\Sigma(v)$ is found by multiplying Eq. (40) by v_k. This yields

$$2\pi\int dv'\frac{v'^2}{v_T^2}f_{\rm M}(v')v^2v'^2\left[\Phi'(x) - \Phi'(x')\right] \times \int_{-1}^1 dy\frac{1-y^2}{\left(v^2+v'^2-2vv'y\right)^{3/2}} = \sum(x)v^2, \quad (41)$$

where $y = \cos\theta$ and θ is the angle between $\mathbf{v}$ and $\mathbf{v}'$. Integration over the angles gives

$$\int_{-1}^{1} dy \frac{1-y^2}{\left(v^2+v'^2-2vv'y\right)^{3/2}}$$

$$= \frac{2}{3v'^3v^3}\left[v^3+v'^3-\left(v^2+vv'+v'^2\right)\left|v-v'\right|\right]$$

$$= \begin{cases} 4/\left(3v^3\right) & \text{if } v' < v \\ 4/\left(3v'^3\right) & \text{if } v' > v \end{cases}. \tag{42}$$

Substituting Eq. (42) into Eq. (41) yields

$$\Sigma(x) = \frac{4n}{3v_T^3\sqrt{2\pi}x^{3/2}}\overline{\Sigma}(x), \tag{43}$$

$$\overline{\Sigma}(x) = \int_0^x dxx'^{3/2}e^{-x'}\left[\Phi'(x)-\Phi'(x')\right]$$

$$+ x^{3/2}\int_x^\infty dx'e^{-x'}\left[\Phi'(x)-\Phi'(x')\right]. \tag{44}$$

Thus, the electron–electron collision integral reduces to

$$\delta J_{\mathrm{ee}}\left[\Phi(x)\right] = \frac{\nu_{\mathrm{ei}}}{\overline{Z}f_{\mathrm{M}}}\partial_{v_k}\left[v_k f_{\mathrm{M}}(v)\frac{\overline{\Sigma}(x)}{x^{3/2}}\right]. \tag{45}$$

The next step is to substitute Eq. (45) into Eq. (35) and solve the latter for Φ. To simplify the integration, the right-hand side of Eq. (35) can be rewritten in the form

$$\phi(x) = \frac{1}{f_{\mathrm{M}}}\partial_{v_k}\left[v_k f_{\mathrm{M}}\frac{\overline{\phi}(x)}{x^{3/2}}\right]. \tag{46}$$

Then, integrating Eq. (35) once, the following integro-differential equation is obtained:

$$\overline{\phi} = \frac{\nu_{\mathrm{ei}}}{\overline{Z}}\overline{\Sigma}(x), \tag{47}$$

where function $\overline{\phi}$ is related to ϕ by integrating Eq. (46),

$$\overline{\phi} = \frac{e^x}{2}\int_\infty^x dx'\phi(x')\sqrt{x'}e^{-x'}. \tag{48}$$

To solve Eq. (47) we take the x derivative of both sides of Eq. (47). This gives

$$\frac{2\overline{Z}}{3\nu_{\mathrm{ei}}}\overline{\phi}' = \Phi''\gamma\left(\frac{3}{2},x\right) - \sqrt{x}\int_x^\infty dx'e^{-x'}\Phi''(x'), \tag{49}$$

where $\gamma(\alpha,x) = \int_0^x x'^{\alpha-1}e^{-x'}dx'$ is the incomplete gamma function. Introducing a new function $g(x) = \int_x^\infty dx'\Phi''(x)e^{-x'}$, Eq. (49) becomes

$$g(x) = -\frac{2\overline{Z}}{3\nu_{\mathrm{ei}}\gamma(3/2,x)}\int_0^x dx'\overline{\phi}'(x')e^{-x'} + \frac{C}{\gamma(3/2,x)}, \tag{50}$$

where C is the integration constant. Thus, the function $\Phi(x)$ can be expressed as a multiple integral of $\overline{\phi}$:

$$\Phi(x) = \overline{C}_1 + \overline{C}_2 x - \int dx'\int dx''e^{x''}g'(x''). \tag{51}$$

Next, we report the equations corresponding to the function of the second and third types [Eqs. (36) and (37), respectively]. Equation (36) reduces to

$$\frac{x^{3/2}}{3}\phi(x) + \frac{\sqrt{\pi}\nu_{\mathrm{ei}}}{4}\Phi(x) = \frac{\nu_{\mathrm{ei}}}{\overline{Z}}\left\{\Phi(x)\left[x^{3/2}e^{-x} - \gamma\left(\frac{3}{2},x\right)\right]\right.$$

$$+ \Phi'(x)\left[(1-x)\gamma\left(\frac{3}{2},x\right) + x^{3/2}e^{-x}\right] + \Phi''(x)x\gamma\left(\frac{3}{2},x\right)$$

$$+ \int_0^x dx'e^{-x'}x'^{3/2}\left(\frac{2x'}{5}-\frac{1}{3}\right)\Phi(x')$$

$$\left. + x^{3/2}\left(\frac{2x}{5}-\frac{1}{3}\right)\int_x^\infty dx'\Phi(x')e^{-x'}\right\}. \tag{52}$$

Equation (37) for the function of the third type becomes

$$\frac{2x^{3/2}}{3}\phi(x)+\frac{3\sqrt{\pi}}{2}\nu_{\mathrm{ei}}\Phi(x)$$

$$=\frac{\nu_{\mathrm{ei}}}{\overline{Z}}\left\{\Phi(x)\left[\gamma\left(\frac{5}{2},x\right)\left(\frac{1}{x}-\frac{10}{3}\right)-\frac{4}{3}x^{3/2}e^{-x}\right]\right.$$

$$+\ 2\Phi'(x)\left[\gamma\left(\frac{3}{2},x\right)(2-x)+x^{3/2}e^{-x}\right]+2x\Phi''(x)\gamma\left(\frac{3}{2},x\right)$$

$$+\ \frac{2}{x}\int_0^x dx'x'^{5/2}e^{-x'}\Phi(x')\left(\frac{12}{35}x'-\frac{2}{15}x-\frac{1}{5}\right)$$

$$\left.+\ 2x^{3/2}\int_x^\infty dx'e^{-x'}\left(\frac{12}{35}x-\frac{2}{15}x'-\frac{1}{5}\right)\Phi(x')\right\}. \tag{53}$$

The heat transport coefficients in the first approximation depend on functions Φ_{13} and Φ_{14}, which belong to the function of the second type and can be found by solving Eq. (52) with

$$\phi(x)=\left(x-\frac{5}{2}\right),\quad \text{for}\quad \Phi=\Phi_{13}/\nu_{\mathrm{ei}}, \tag{54}$$

$$\phi(x)=1,\quad \text{for}\quad \Phi=\Phi_{14}/\nu_{\mathrm{ei}}. \tag{55}$$

To solve the integro-differental equation (52), function $\Phi(x)$ is traditionally expanded[10,11] in Laguerre polynomials[12] $\Phi(x)=\sum_n A(n)L_n^{3/2}$. As proposed in Ref. 9, it is more convenient to use a more-generalized expansion in terms of Laguerre polynomials $L_n^\alpha(x)$. The choice of these polynomials comes from their orthogonal properties

$$\int_0^\infty e^{-x}x^\alpha L_m^\alpha(x)L_n^\alpha(x)dx$$

$$=\begin{cases}0 & \text{if } m\neq n\\ \Gamma(n+\alpha+1)/n! & \text{if } m=n,\ \alpha>-1,\quad n=0,\ 1,\ 2,\ldots\ .\end{cases} \tag{56}$$

Evaluation of the integrals in Eqs. (32) and (33) becomes particularly simple if

$$\Phi_{13(14)}=x^\beta\sum_n A(n)L_n^{\beta+3/2}. \tag{57}$$

Index β is determined by matching the polynomial expansion (57) with the exact solution of Φ in the limit of $\overline{Z}\to\infty$. Calculations show that such matching speeds up the convergence of the transport coefficients with the number of polynomials in expansion (57). Taking the limit $\overline{Z}\to\infty$ in Eq. (52) yields

$$\Phi\big|_{\overline{Z}\to\infty}=-\frac{4}{3\sqrt{\pi}\nu_{\mathrm{ei}}}x^{3/2}\phi. \tag{58}$$

Then, the choice $\beta = 3/2–k$ with $k = 0, 1, 2,\ldots$ will satisfy the requirement of matching Eq. (57) with the exact solution (58). The parameter k is determined by minimizing the number of terms in the polynomial expansion (1) to match the exact solution for $\overline{Z}\to\infty$ and (2) to reach the desired accuracy of the transport coefficients for $\overline{Z}\sim 1$. Calculations show that for the case of functions Φ_{13} and Φ_{14}, $\beta = 1/2$ satisfies such a minimization criteria [it takes five terms in Eq. (57) to obtain the transport coefficients with 1% accuracy]. Therefore, the expansion becomes

$$\Phi_{13(14)}=\sqrt{x}\sum_n A_{13(14)}(n)L_n^2. \tag{59}$$

Multiplying Eq. (52) by $x^{3/2}e^{-x}L_s^\alpha(x)$ with $s = 0, 1, 2,\ldots, N–1$ [where N is the number of polynomials in the expansion (57)] and integrating the latter in x from 0 until ∞, we obtain the system of N algebraic equations. Figure 98.4 shows a dependence of the coefficients α_{j1}^T and α_{q1}^T on the number of polynomials in the expansion (57) with $\beta = –1/2$, $\beta = 1/2$, and $\beta = 3/2$, respectively. Observe that the coefficients converge faster with $\beta = 1/2$.

Next, we derive the numerical coefficient α_σ^E of the stress tensor $\sigma_{ij}^{(1)}$. This requires that Eq. (53) be solved with $\Phi(x) = \Phi_{11}$ and

$$\phi=-\frac{3\nu_{\mathrm{ei}}}{8x^{3/2}}\left\{\frac{\sqrt{\pi}}{2}\left(1+\frac{3}{2x}\right)+\frac{3}{\overline{Z}}\left[\gamma\left(\frac{3}{2},x\right)-\frac{1}{x}\gamma\left(\frac{5}{2},x\right)\right]\right\}. \tag{60}$$

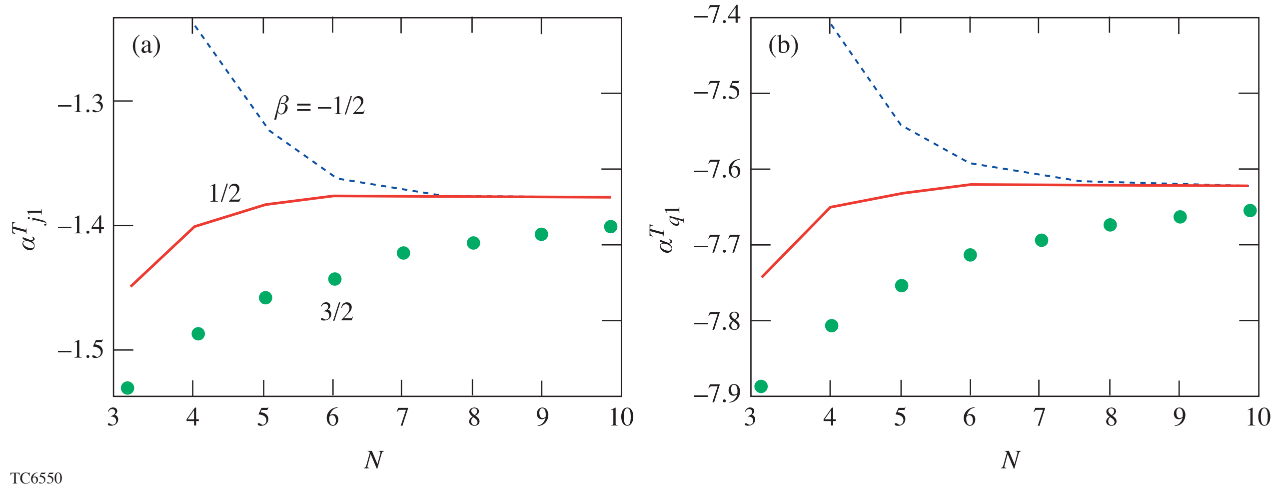

Figure 98.4
Coefficients α^T_{j1} and α^T_{q1} as functions of the number of polynomials in the expansion (57). The results correspond to $\beta = -1/2$ (dashed line), $\beta = 1/2$ (solid line), and $\beta = 3/2$ (dots).

Similar to the previously considered case, we expand function Φ_{11} in Laguerre polynomials. To express α^E_σ through just one coefficient in such an expansion, we take

$$\Phi_{11} = x^{\beta_1} \sum_n B(n) L_n^{\beta_1 + 5/2}. \tag{61}$$

The choice of the power index β_1 comes from the condition of matching expansion (61) with the exact solution in the limit of $\bar{Z} \to \infty$. Neglecting terms proportional to $1/\bar{Z}$ in Eqs. (53) and (60) gives

$$\Phi_{11}(x)\big|_{\bar{Z}\to\infty} = \frac{1}{8x} + \frac{1}{12}. \tag{62}$$

It is easy to see that the values $\beta_1 = -1, -2, -3,\ldots$ satisfy our requirement. Calculations show that expansion (61) with $\beta_1 = -1$ has the fastest convergence with the number of polynomials. Table 98.II shows a summary of coefficients α^E_σ for a different ion charge $\bar{Z}$. Observe that the stress tensor has a very weak dependence on $\bar{Z}$ (3% variation in α^E_σ from $\bar{Z} = 1$ to $\bar{Z} = \infty$). One more function remains to be determined in the first approximation: the correction Φ_{12} to the symmetric part of the distribution function. This function belongs to the first type and can be found in the integral form using Eq. (51) with

$$\phi(x) = \left(\frac{3}{2} - x + \frac{3\sqrt{\pi}}{4\sqrt{x}}\right)\frac{\nu_{ei}}{3} \tag{63}$$

Table 98.II: Transport coefficients in the first approximation.

$\bar{Z}$	1	2	3	4	5	10	30	80	∞
α^T_{j1}	–1.39	–2.1	–2.57	–2.91	–3.16	–3.87	–4.59	–4.89	–5.09
α^T_{q1}	–7.66	–12.11	–15.19	–17.46	–19.23	–24.31	–29.86	–32.27	–33.95
α^T_{j2}	–1.99	–2.34	–2.54	–2.67	–2.77	–3.01	–3.25	–3.34	–3.39
α^T_{q2}	–6.35	–7.93	–8.90	–9.57	–10.07	–11.40	–12.70	–13.23	–13.58
α^E_σ	1.029	1.027	1.023	1.020	1.017	1.010	1.004	1.002	1.000

and $\Phi = \Phi_{12}$. The integration gives[8]

$$\Phi_{12}(x) = \bar{Z}\left\{-\tilde{C}_2 + C_2 x - \frac{\sqrt{\pi}}{12}\int_1^x \frac{dt\sqrt{t}(x-t)}{\left[\gamma(3/2,t)\right]^2}\right\}, \qquad (64)$$

where C_2 and $\tilde{C}_2$ are determined from the condition of zero contribution of Φ_{12} to the electron density and temperature,

$$\int_0^\infty dx e^{-x}\sqrt{x}\,\Phi_{12}(x) = 0, \quad \int_0^\infty dx e^{-x} x^{3/2}\Phi_{12}(x) = 0. \qquad (65)$$

Conditions (65) yield $C_2 = 0.721$ and $\tilde{C}_2 = 0.454$. Note two misprints in Φ_{12} reported in Ref. 8 [the different sign in front of the integral and $(x - t)$ instead of $(1 - t)$ inside the integral]. The correction to the symmetric part in the distribution function comes mainly from balancing the inverse bremsstrahlung heating $\sqrt{\pi/x}\,\nu_{ei}/3\left(v_E^2/v_T^2\right)$ with the electron–electron collisions δJ_{ee}. Since $\delta J_{ee} \sim \nu_{ei}/\bar{Z}$, function Φ_{12} becomes proportional to the average ion charge $\bar{Z}$, as shown in Eq. (64). As emphasized in Ref. 8, the symmetric correction Φ_{12} gives the dominant contribution to the heat flux in the second-order approximation.

Second-Order Approximation

Correction ψ_2 to the distribution function in the second approximation satisfies the following equation:[8]

$$\left(x-\frac{5}{2}\right)\frac{\nabla \mathbf{j}^{(1)}}{en} - \left(x-\frac{3}{2}\right)\frac{2\nabla \mathbf{q}^{(1)}}{3nT} + \frac{v^2}{3\nu_{ei}}\left[\Phi_{13}\nabla^2 \ln T\right.$$

$$\left. + \Phi_{14}\nabla\left(\nabla \ln nT - \frac{e\mathbf{E}_0}{T}\right)\right]$$

$$+ \mathbf{v}\frac{\nabla v_E^2}{v_T^2}\left[\Phi_{12} + \frac{1}{3}\left(\Phi_{13}+\Phi_{14}\right) + \frac{1}{2} - \frac{x}{6}\right]$$

$$+ \frac{v_i \partial_{r_k}\left(\mathbf{v}_E^2\right)_{ik}}{10 v_T^2}\left(8x\Phi_{11} - x\right) - \frac{e}{\nu_{ei}T}\Phi_{14}\mathbf{v}\partial_t \mathbf{E}_0$$

$$+ \frac{\left(\mathbf{v}^2\right)_{ij}}{\nu_{ei}}\left[\Phi_{13}\frac{\partial^2 \ln T}{\partial r_i \partial r_j} + \Phi_{14}\partial_{r_i}\left(\partial_{r_j}\ln nT - \frac{eE_{0j}}{T}\right)\right]$$

$$+ \left(\mathbf{v}^3\right)_{ijk}\frac{\partial_{r_k}\left(\mathbf{v}_E^2\right)_{ij}}{v_T^4}\left(\Phi_{11} - \frac{1}{8}\right) = \delta J_{ei}\left[\psi_2\right] + \delta J_{ee}\left[\psi_2\right]. \qquad (66)$$

A general solution of Eq. (66) can be written as

$$\psi_2 = \Phi_{20}\frac{\left(\mathbf{v}^3\right)_{ijk}\partial_{r_k}\left(\mathbf{v}_E^2\right)_{ij}}{v_T^4 \nu_{ei}} + \Phi_{21}\frac{v_i \partial_{r_j}\left(\mathbf{v}_E^2\right)_{ij}}{\nu_{ei} v_T^2}$$

$$+\Phi_{22}\frac{\mathbf{v}\nabla v_E^2}{v_T^2 \nu_{ei}} + \frac{\left(\mathbf{v}^2\right)_{kj}}{\nu_{ei}^2}\left\{\Phi_{23}\left(\frac{\partial^2 \ln T}{\partial r_k \partial r_j} - \frac{1}{3}\delta_{jk}\nabla^2 \ln T\right)\right.$$

$$\left. +\Phi_{24}\left[\frac{\partial}{\partial_{r_k}}\left(\partial_{r_j}\ln nT - \frac{eE_{0j}}{T}\right) - \frac{1}{3}\delta_{jk}\nabla\left(\nabla \ln nT - \frac{e\mathbf{E}_0}{T}\right)\right]\right\}$$

$$-\Phi_{25}\frac{e}{\nu_{ei}T}\mathbf{v}\partial_t \mathbf{E}_0$$

$$+\frac{v_T^2}{\nu_{ei}^2}\left[\Phi_{26}\nabla^2 \ln T + \Phi_{27}\nabla\left(\nabla \ln nT - \frac{e\mathbf{E}_0}{T}\right)\right]. \qquad (67)$$

The electric current and the heat flux in the second order take the form

$$j_i^{(2)} = env_T\lambda_e\left[\alpha_{j1}^E\frac{\partial_{r_k}\left(\mathbf{v}_E^2\right)_{ik}}{v_T^2} + \alpha_{j2}^E\frac{\partial_{r_i}v_E^2}{v_T^2}\right.$$

$$\left. -\alpha_{j3}^E\frac{e\lambda_e}{v_T T}\partial_t E_{0i}\right], \qquad (68)$$

$$q_i^{(2)} = nTv_T\lambda_e\left[\alpha_{q1}^E\frac{\partial_{r_k}\left(\mathbf{v}_E^2\right)ik}{v_T^2} + \alpha_{q2}^E\frac{\partial_{r_i}v_E^2}{v_T^2}\right.$$

$$\left. -\alpha_{q3}^E\frac{e\lambda_e}{v_T T}\partial_t E_{0i}\right]. \qquad (69)$$

Coefficients $\alpha^E_{j(q)}$ are calculated using the following relations:

$$\alpha^E_{j1(2,3)} = \frac{4}{3\sqrt{\pi}} \int_0^\infty dx x^{3/2} e^{-x} \Phi_{21(2,5)}(x), \tag{70}$$

$$\alpha^E_{q1(2,3)} = \frac{4}{3\sqrt{\pi}} \int_0^\infty dx x^{5/2} e^{-x} \Phi_{21(2,5)}(x). \tag{71}$$

Next, we find functions Φ_{21}, Φ_{22}, and Φ_{25}. These functions are of the second type; therefore, to obtain them we solve Eq. (52) with

$$\phi = \frac{x}{10}\left(8\Phi_{11} - 1\right), \quad \text{for } \Phi = \Phi_{21}/\nu_{\text{ei}}, \tag{72}$$

$$\phi = \Phi_{12} + \frac{1}{3}\left(\Phi_{13} + \Phi_{14}\right) + \frac{1}{2} - \frac{x}{6}, \quad \text{for } \Phi = \Phi_{22}/\nu_{\text{ei}}, \tag{73}$$

$$\phi = \Phi_{14}, \quad \text{for } \Phi = \Phi_{25}/\nu_{\text{ei}}. \tag{74}$$

Following the method described in the previous section, functions Φ_{21}, Φ_{22}, and Φ_{25} are expanded in series (57). The exact solution for Φ_{21} as $\bar{Z} \to \infty$ becomes

$$\Phi_{21}\big|_{\bar{Z}\to\infty} = -\frac{2x^{3/2}}{15\sqrt{\pi}}\left(1 - \frac{x}{3}\right); \tag{75}$$

thus β takes the values $\beta = 3/2 - k$ with $k = 0, 1, 2, \ldots$. The fastest convergence of the coefficients α^E_{j1} and α^E_{q1} is obtained with $\beta = 1/2$. A summary of α^E_{j1} and α^E_{q1} for different ion charge $\bar{Z}$ is given in Table 98.III. Next, we find the function Φ_{22}. The exact matching of the polynomial expansion (57) with the exact solution Φ_{22} for $\bar{Z} \to \infty$,

$$\Phi_{22}\big|_{\bar{Z}\to\infty} = -\frac{4x^{3/2}}{3\sqrt{\pi}}\Phi_{12}, \tag{76}$$

cannot be done since Φ_{12} does not have a polynomial structure [see Eq. (64)]. It is easy to show, however, that $\Phi_{12}(x \to 0) \sim 1/\sqrt{x}$ and $\Phi_{12}(x \to \infty) \sim x^{5/2}$. Therefore, the expansion of Φ_{22} with $\beta = 1$ reproduces the asymptotic limits for $x << 1$ and $x >> 1$. Taking $\beta = 1$ and keeping $N = 5$ terms in expansion (57) gives values of α^E_{j2} and α^E_{q2}, which are reported in Table 98.III. Observe that these coefficients become quite large for $\bar{Z} >> 1$. To find the remaining coefficients in the heat flux and electric current, we solve the equation for the function Φ_{25}. In the limit of $\bar{Z} \to \infty$, the function Φ_{25} becomes

$$\Phi_{25}\big|_{\bar{Z}\to\infty} = -\Phi_{14}\big|_{\bar{Z}\to\infty} \frac{4x^{3/2}}{3\sqrt{\pi}} = \frac{16}{9\pi}x^3; \tag{77}$$

thus, β = 3, 2, 1… matches the polynomial expansion (57) with the exact solution in the limit of $\bar{Z} \to \infty$. Calculations show that $\beta = 1$ requires a minimum number of polynomials in expansion (57) to achieve the desired accuracy. The values of α^E_{j3} and α^E_{q3} are summarized in Table 98.III. Next, we

Table 98.III: Transport coefficients in the second approximation.

$\bar{Z}$	1	2	3	4	5	10	30	80	∞
α^E_{j1}	–0.03	–0.01	0.00	0.02	0.03	0.06	0.09	0.10	0.11
α^E_{q1}	–0.03	0.07	0.16	0.23	0.30	0.49	0.73	0.83	0.90
α^E_{j2}	4.05	8.54	13.07	17.51	21.86	42.30	116.3	$\bar{Z}$ 3.66	$\bar{Z}$ 3.48
α^E_{q2}	19.7	48.3	80.0	113.0	146.6	314.1	960.9	$\bar{Z}$ 31.7	$\bar{Z}$ 31.3
α^E_{j3}	4.69	7.21	9.07	10.51	11.67	15.15	19.18	21.00	22.28
α^E_{q3}	16.77	28.75	38.45	46.39	52.99	74.05	100.6	113.3	122.5

combine the electric current and the thermal flux in the first and second approximations. The result is

$$\mathbf{j}_i = env_T\lambda_e\left[\alpha_{j1}^T\partial_{r_i}\ln T + \alpha_{j2}^T\left(\partial_{r_i}\ln nT - \frac{eE_{0i}}{T}\right)\right.$$

$$\left. + \alpha_{j1}^E\frac{\partial_{r_k}\left(\mathbf{v}_E^2\right)_{ik}}{v_T^2} + \alpha_{j2}^E\frac{\partial_{r_i}v_E^2}{v_T^2} - \alpha_{j3}^E\frac{e\lambda_e}{v_T T}\partial_t E_{0i}\right], \qquad (78)$$

$$\mathbf{q}_i = nTv_T\lambda_e\left[\alpha_{q1}^T\partial_{r_i}\ln T + \alpha_{q2}^T\left(\partial_{r_i}\ln nT - \frac{eE_{0i}}{T}\right)\right.$$

$$\left. + \alpha_{q1}^E\frac{\partial_{r_k}\left(\mathbf{v}_E^2\right)_{ik}}{v_T^2} + \alpha_{q2}^E\frac{\partial_{r_i}v_E^2}{v_T^2} - \alpha_{q3}^E\frac{e\lambda_e}{v_T T}\partial_t E_{0i}\right]. \qquad (79)$$

Imposing a condition of zero current $\mathbf{j} = 0$ and also assuming $t_E\nu_{\mathrm{ei}} << \alpha_{j3}^E/\alpha_{j2}^T$ (where t_E is the time scale of $\mathbf{E}_0$ variation) define the slowly varying component of the electric field $\mathbf{E}_0$,

$$\frac{eE_{0i}}{T} = \partial_{r_i}\ln nT + \frac{\alpha_{j1}^T}{\alpha_{j2}^T}\partial_{r_i}\ln T + \frac{\alpha_{j1}^E}{\alpha_{j2}^T}\frac{\partial_{r_k}\left(\mathbf{v}_E^2\right)_{ik}}{v_T^2}$$

$$+ \frac{\alpha_{j2}^E}{\alpha_{j2}^T}\frac{\partial_{r_i}v_E^2}{v_T^2}. \qquad (80)$$

Substituting $\mathbf{E}_0$ from Eq. (80) into Eq. (79) gives the heat flux in laser-produced plasmas,

$$\mathbf{q}_i = nTv_T\lambda_e\left[\beta^T\partial_{r_i}\ln T + \beta_1^E\frac{\partial_{r_i}v_E^2}{v_T^2} + \beta_2^E\frac{\partial_{r_k}\left(\mathbf{v}_E^2\right)_{ik}}{v_T^2}\right], \qquad (81)$$

where

$$\beta^T = \alpha_{q1}^T - \alpha_{q2}^T\alpha_{j1}^T/\alpha_{j2}^T,$$

$$\beta_1^E = \alpha_{q2}^E - \alpha_{q2}^T\alpha_{j2}^E/\alpha_{j2}^T,$$

and

$$\beta_2^E = \alpha_{q1}^E - \alpha_{q2}^T\alpha_{j1}^E/\alpha_{j2}^T$$

can be represented with the following fitting formulas:

$$\beta^T = -\frac{128}{3\pi}\frac{\overline{Z}+0.24}{\overline{Z}+4.20},$$

$$\beta_1^E = 17.31\,\overline{Z}\,\frac{\overline{Z}^2+14.04\,\overline{Z}+2.41}{\overline{Z}^2+14.34\,\overline{Z}+29.5}, \qquad (82)$$

$$\beta_2^E = 0.45\frac{\overline{Z}-0.29}{\overline{Z}+3.47}. \qquad (83)$$

In addition, the coefficients in the electric field E_0 can be fitted as follows:

$$\frac{\alpha_{j1}^T}{\alpha_{j2}^T} = 1.50\;\frac{\overline{Z}+0.52}{\overline{Z}+2.26},$$

$$\frac{\alpha_{j1}^E}{\alpha_{j2}^T} = -\;0.03\;\frac{\overline{Z}-2.63}{\overline{Z}+3.22}, \qquad (84)$$

$$\frac{\alpha_{j2}^E}{\alpha_{j2}^T} = -1.03\,\overline{Z}\,\frac{\overline{Z}+8.54}{\overline{Z}+3.82}.$$

Coefficients β^T and $\alpha_{j1}^T/\alpha_{j2}^T$ agree with previously published results.[1,11,13]

Next, we discuss the validity of the derived transport coefficients. As shown earlier, the main contribution to the second-order heat flux comes from the correction Φ_{12} to the symmetric part of the distribution function. The function Φ_{12} is given in the integral form by Eq. (64) and has the following asymptotic behavior for small and large velocities:

$$\Phi_{12}(x \to 0) = -\frac{\bar{Z}}{4}\sqrt{\frac{\pi}{x}}, \tag{85}$$

$$\Phi_{12}(x \to \infty) = -\bar{Z}\frac{4}{45\sqrt{\pi}}x^{5/2}. \tag{86}$$

The validity condition of the Chapman–Enskog method[10] $|\Phi_{12}|v_E^2/v_T^2 << 1$ breaks down for

$$x < \bar{Z}^2\frac{\pi}{16}\frac{v_E^4}{v_T^4} \tag{87}$$

and

$$x > \frac{3}{\bar{Z}^{2/5}}\left(\frac{v_T}{v_E}\right)^{4/5}. \tag{88}$$

According to Eq. (71), the main contribution to the heat flux comes from the superthermal electrons [which correspond to the maximum in the function $x^{5/2}e^{-x}\Phi(x)$]. Therefore, the limit (87) imposes no restrictions on the applicability of the derived results. The electron distribution function for the subthermal electrons, nevertheless, is different from the limit (85). As derived in Refs. 14 and 15, the inverse bremsstrahlung heating modifies the distribution of the cold electrons to

$$f_0(v << v_T) = \frac{n}{(2\pi)^{3/2}v_T^3}\exp\left(-\frac{1}{v_T^2}\int_0^v\frac{u^4du}{u^3+V_L^3}\right), \tag{89}$$

where $V_L = \left(\sqrt{\pi/8}\,\bar{Z}v_E^2v_T\right)^{1/3}$ is the Langdon velocity.[16] To check the limitations due to the second condition (88), we find that the maximum of

$$x^{5/2}e^{-x}\Phi_{22} \sim x^{5/2}e^{-x}x^{3/2}\Phi_{12} \sim x^{13/2}e^{-x}$$

corresponds to $x_{\max} \simeq 13/2$. This limits the applicability of the Chapman method to $v_E^2/v_T^2 < 0.2/\bar{Z}$. Even though the modulus of Φ_{12} becomes larger than unity for large x [see Eq. (88)], we can show that

$$f_0^{\mathrm{int}} = f_{\mathrm{M}}\exp\left[1+\frac{v_E^2}{v_T^2}\Phi_{12}(x)\right] \tag{90}$$

is a good approximation to the symmetric part of the distribution function even for $x \to \infty$. For such a purpose, we find the asymptotic behavior of the function that satisfies the following equation:

$$\partial_t f_0 = J_{\mathrm{ee}}(f_0, f_0). \tag{91}$$

We look for a solution of Eq. (91) in the form $f_0 = Ae^{\Psi}$, where $\Psi = F(v_T^2)g(v^2)$ and A is a normalization constant. The temperature dependence is combined in function F, and velocity dependence is in g; then, the time derivative of f_0 becomes

$$\partial_t f_0 = f_0F'v_T^2g\frac{\partial_t T}{T} = f_0F'g\frac{v_E^2\nu_{\mathrm{ei}}}{3}, \tag{92}$$

where we substituted $\partial_t T/T = \left(v_E^2/v_T^2\right)\nu_{\mathrm{ei}}/3$ due to the inverse bremsstrahlung heating. The electron–electron collision integral reduces in this case to

$$J_{\mathrm{ee}} = \frac{16\pi e^4\Lambda}{3m^2v}F\frac{\partial}{\partial v^2}\left[f_0I\left(v^2\right)\right], \tag{93}$$

$$I\left(v^2\right) = 4\pi\int_0^v dv'v'^4f_0(v')\left[g'\left(v^2\right)-g'\left(v'^2\right)\right]$$
$$+\ 4\pi v^3\int_v^\infty dv'v'f_0(v')\left[g'\left(v^2\right)-g'\left(v'^2\right)\right]. \tag{94}$$

In the limit of $v \to \infty$, I becomes

$$I \simeq g'\left(v^2\right)4\pi\int_0^\infty dv'v'^4f_0(v') = 3g'\left(v^2\right)nv_T^2,$$

and Eq. (91) takes the form

$$g'' + Fg'^2 = \bar{Z}\frac{vv_E^2}{v_T^5}\frac{F'}{F}\frac{g}{18\sqrt{2\pi}}. \tag{95}$$

Next, we make an assumption $g'' << g'^2$, which will be verified *a posteriori*. In this case the solution of Eq. (95) becomes

$$g = \frac{2}{225\sqrt{2\pi}}\, v^5 \frac{F'}{F^2}\frac{\bar{Z} v_E^2}{v_T^5}. \tag{96}$$

Observe that the condition $g'' << g'^2$ is satisfied in the limit of large velocity. The function g, by definition, does not depend on temperature; this yields for F

$$\frac{F'}{F^2} = C\left(v_T^2\right)^{5/2}, \quad F = \frac{\tilde{C}}{v_T^7}, \tag{97}$$

where C and $\tilde{C}$ are constants. The distribution function f_0 depends on the product $F\left(v_T^2\right)g\left(v^2\right)$, which, according to Eqs. (96) and (97), takes the form

$$F\left(v_T^2\right)g\left(v^2\right) = -\frac{7\bar{Z}}{225\sqrt{2\pi}}\frac{v_E^2 v^5}{v_T^7}. \tag{98}$$

Using Eq. (98), the asymptotic limit of the symmetrical part of the distribution function reduces to

$$f_0(x >> 1) \sim \exp\left(-0.07\,\bar{Z}\,\frac{v_E^2}{v_T^2}x^{5/2}\right). \tag{99}$$

The latter equation must be compared to f_0^{int} in the limit $x \to \infty$ [see Eq. (90)],

$$f_0^{\text{int}}(x >> 1) \sim \exp\left(-0.05\,\bar{Z}\,\frac{v_E^2}{v_T^2}x^{5/2}\right). \tag{100}$$

Thus, we can conclude that the function in the form (90) is a good approximation to the distribution function for thermal and superthermal electrons.

In conclusion, we have derived the transport coefficients, including the thermal and ponderomotive terms for an arbitrary ion charge. The modification of the thermal transport due to the ponderomotive effects near the critical surface and laser turning point will be discussed in a forthcoming publication.

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