Hydrodynamics of Inertial Fusion Capsules: Feedout and Deceleration Phase Instability.

by

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Abstract

In Inertial Confinement Fusion (ICF), a spherical shell is accelerated inward by laser irradiation, and the shell outer surface is unstable to the Rayleigh-Taylor (RT) instability. During the acceleration phase, the instability can be seeded by both inner and outer surface nonuniformities. The seeding of the instability by the inner surface nonuniformities (typically referred to as “feedout”) is investigated both analytically and numerically. A simple formula relating the outer surface distortion to the inner surface roughness is derived for long wavelength perturbations satisfying the condition $kd < 1$, where $k$ is the perturbation wavenumber, $d$ is the shell thickness. The validity of the analytic feedout formula derived for long wavelength modes is extended to the short wavelengths by fitting the results of two-dimensional Lagrangian simulation.

When the laser is turned off and the shell is decelerated by the large pressure building up in the center of the capsule, the shell inner surface is RT unstable. The dynamics of the shell and hot-spot during the deceleration phase is investigated both numerically and analytically. It is shown that mass ablation off the shell inner surface significantly reduces the growth rates of the deceleration phase Rayleigh-Taylor instability. It is also found that for typical direct drive capsules designed for the National Ignition Facility, the instability of Legendre modes with $l > 90$ is suppressed by the ablative stabilization.
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Chapter 1

Introduction.

In inertial confinement fusion (ICF, see [1, 2]), a spherical shell (the “capsule”) of cryogenic deuterium and tritium (DT) filled with DT gas is accelerated inward by direct laser irradiation (“direct drive”, figure 1.1) or X-rays produced by a laser irradiated high-Z enclosure (“indirect drive”). The laser pulse begins with a constant, low-intensity foot designed to drive an ionizing shock wave through the capsule. While converging to the center of the capsule, the shock heats up the ionized gas to temperatures of the order of a few keVs. Such a low density hot plasma is commonly referred to as the “hot-spot”. While the shock is propagating in the gas, the laser power rises and the shell is accelerated inward. The hot-spot is heated up further, reaching thermonuclear temperatures, by the compression work provided by the incoming shell.

For a DT fuel, the primary thermonuclear reaction produces a 3.5 MeV alpha particle ($\text{He}_4$) and a 14 MeV neutron:

$$D + T \rightarrow \text{He}_4 + n$$ (1.1)

While the energetic neutrons leave the capsule, the alpha particles are slowed down.
Figure 1.1: The target is irradiated by laser beams and accelerated inwards due to the mass ablation off the surface.

by collisions with the surrounding electrons and release part or all their energy to the plasma ("thermonuclear heating"). Typically, a significant fraction of alpha particles is slowed down within the hot-spot when the alpha mean free path is less than the hot-spot radius. When the hot-spot material has reached temperatures of about 10\text{keV} and areal densities of about 0.3\text{g/cm}^2, the thermonuclear heating is sufficient to overcome all the energy losses from the hot-spot leading to a thermal instability and a rapid increase in temperatures up to 20 ÷ 30\text{keV}. The onset of such a thermal instability is commonly referred to as "thermonuclear ignition" and leads to an almost complete burn of the hot-spot material on a picoseconds time scale.

Because of the high temperatures and relatively low densities, a fraction of the alpha particles escapes the hot-spot and is deposited on the shell inner surface. Furthermore, a large conductive heat flux leaving the ignited hot-spot is also absorbed by the shell material near the inner surface. Both these effects cause a
rapid heating of the shell surface up to thermonuclear temperatures triggering the propagation of a burn wave through the dense shell. The burn wave is supersonic (see [3]) and heats up the whole shell to thermonuclear temperatures. At this point the burning shell starts expanding outward on a time scale proportional to the time of the sound wave propagation through the shell. As the shell expands, it cools down, disassembles and the thermonuclear burn comes to a stop. If the shell areal density is larger than \( \sim 1 \text{gr/cm}^2 \), the disassembling time is sufficiently large that a significant fraction of the shell material (larger than 10\%) is burned, leading to a substantial energy release in the form of neutron energy.

1.1 Acceleration and Deceleration Phase of ICF Implosions.

In both direct and indirect drive ICF, the laser pulse starts from a constant, low-intensity foot designed to drive a uniform shock through the shell. After the shock breaks out on the shell inner surface, it expands inward launching a shock in the gas and a rarefaction wave in the shell. As the rarefaction wave travels across the shell, the shell outer surface moves at approximately constant velocity. When the rarefaction wave reaches the shell outer surface, the latter starts accelerating and the so called "acceleration phase" begins. At about the shock break-out time, the laser power begins to rise first slowly and then more rapidly in order to keep the shell close to the shock front traveling inside the gas. A second shock, originating within the shell, is launched during the initial pulse rise and merges with the first shock before reaching the center of the capsule. The acceleration phase ends when the laser is turned off and the shell starts traveling at approximately constant
velocity. Standard direct-drive pulse designs make use of such a sequence of two shocks merging into one. Instead, the latest pulse designs of indirect-drive ICF make use of a sequence of four shocks coalescing into one before reaching the center.

In both direct and indirect drive ICF, the single shock resulting from the multiple shock coalescence travels into the gas in the form of a *strong* shock; i.e. $\Delta P/P_b \gg 1$ where $\Delta P$ is the pressure jump across the shock and $P_b$ is the gas pressure before the shock. Such a shock is reflected off the center of the capsule and subsequently is reflected off the incoming inner shell surface which in turn is impulsively decelerated. The shock reflected off the shell travels towards the center to be reflected there again and subsequently reflected a second time from the shell. At each reflection off the shell, the latter is impulsively decelerated and the shock gets weaker until the pressure jump across the shock front is smaller than the pressure before the shock ($\Delta P/P_b < 1$). We denote the time interval corresponding to the multiple shock reflections as the “impulsive deceleration phase”. Typically, the reflected shock becomes *weak* after the first or second reflection off the shell. At this point the material enclosed by the inner shell surface develops a fairly uniform pressure profile and is referred to as the hot-spot. After the second shock reflection off the shell, the hot-spot is formed and its pressure is high enough for the shell velocity to be lower than the hot-spot sound speed; i.e. the flow is subsonic. When the hot-spot is formed, the shell is decelerated in a continuous (not impulsive) manner and acts like a piston on the hot-spot. This continuous slowing down of the shell up to the stagnation point occurs over a period of a few hundred picoseconds and is referred to as the “continuous deceleration phase”. Figure 1.2 shows the time evolution of the deceleration $g$ of a shell designed for direct drive ignition (see [4]) on the National Ignition Facility (NIF). The time $t = 0$ repre-
Figure 1.2: Time history of the inner surface deceleration obtained in 1D numerical simulations of a NIF-like capsule. \( t = 0 \) is the stagnation time. The impulsive deceleration phase is followed by the continuous deceleration phase.

sents the stagnation point and the continuous deceleration starts at about 200ps before stagnation. During the deceleration phase, the hot-spot pressure, density and temperature keep increasing until reaching the ignition conditions. If the shell is sufficiently dense, the ignited hot-spot triggers a burn wave propagating in the shell. A significant fraction (> 10%) of the shell mass undergoes thermonuclear burn if the shell areal density exceeds 1gr/cm².

If the shell surfaces were perfectly smooth and the implosions were spherically symmetric, thermonuclear ignition and burn could have been easily achieved with lasers in the kilojoule energy range. Unfortunately, the shell is unstable during the implosion and small imperfections of the shell surface grow rapidly leading to the angular distortion, symmetry breaking and reduced compression of the capsule.

In both direct and indirect drive, the shell material is ablating off the surface
into a low density plasma and the rest of the shell is pushed inward according to the conservation of momentum ("rocket effect"). In the frame of reference of the ablation front (i.e. the outer shell surface), the net inertial force is directed from the heavy (compressed) to the light (expanding) material, and the target surface is hydrodynamically unstable, the phenomenon being generally known as the Acceleration Phase Rayleigh-Taylor (APRT) instability.

Later in time, when the shell is slowed down by the rapid increase in the hot-spot pressure, the inertial force in the inner shell surface frame of reference is directed from the shell to the hot-spot, and the inner surface is Rayleigh-Taylor unstable [Deceleration Phase Rayleigh-Taylor (DPRT) instability]. Since thermal energy is lost from the hot-spot through heat conduction to the inner shell surface, the distortion induced by the deceleration phase RT instability can significantly increase the inner surface area and enhance the heat losses from the hot-spot. Furthermore large amplitude surface perturbations can develop into spikes of cold shell material reaching into the center of the hot-spot leading to a mixing of the cold and hot plasma and quenching of the hot spot ignition. The fast growing, short wavelength modes can also reduce the compression work of the shell by channeling inward radial kinetic energy into lateral kinetic energy. In other words, a fraction of the shell energy could be transformed into turbulent kinetic energy by the instability making less shell energy available for the compression work on the hot-spot.

1.2 Rayleigh-Taylor Instability.

During both the acceleration and the deceleration phases of the implosion, the shell is unstable to the Rayleigh-Taylor instability. The latter is the instability
occurring at the interface between two superimposed fluids of different densities in
the presence of a gravitational field directed from the heavy to the light fluid.

![Diagram of Rayleigh-Taylor instability](image)

Figure 1.3: Rayleigh-Taylor unstable configuration. The gravitational $g$ directed from the heavy to the light fluid drives the Rayleigh-Taylor instability of the fluid interface.

If the two fluids are ideal (non dissipative) and incompressible, the instability is commonly known as a “classical” Rayleigh-Taylor instability. In the classical RT instability with constant gravitational field, a small (i.e. linear) sinusoidal perturbation would grow exponentially in time at a rate depending on the gravitational acceleration $g$, the perturbation wavelength $\lambda$ and the fluid densities $\rho_h$, $\rho_l$ where the subscripts $h$ and $l$ indicate the heavy and light fluid respectively. If the two fluids are separated by a sharp interface and their densities are uniform, the exponential linear growth rate of the instability can be written as

$$
\gamma = \sqrt{A_T k g} \tag{1.2}
$$

where $k = 2\pi/\lambda$ is the perturbation wavenumber, and

$$
A_T = \frac{\rho_h - \rho_l}{\rho_h + \rho_l} \tag{1.3}
$$
is the Atwood number (see [5]). It is important to notice that the growth rate increases with the mode wavenumber thus indicating that short wavelength modes have the largest growth rates. However, if the interface between fluids is not sharp and the density profile smoothly changes between the heavy and the light fluid, the growth rate of short wavelength modes tends asymptotically to the constant value (see [6])

$$\gamma = \sqrt{\frac{g}{L_m}}$$

where $L_m = \min |\rho/(d\rho/dy)|$ is the minimum value of the density gradient scale length. Equation 1.4 is a good approximation of the growth rate as long as the mode wavenumber satisfies the condition $kL_m \gg 1$ while the equation 1.2 represents the growth rates for modes with $kL_m \ll 1$.

During the acceleration phase, the outer shell surface is unstable because in its frame of reference, the inertial force is equivalent to a gravitational field pointing from the heavy shell towards the low density ablated plasma. However, the development of the Rayleigh-Taylor instability is quite different from the classical case. In direct drive ICF, the laser energy is absorbed in the coronal plasma at the critical surface where the laser frequency equals the plasma frequency. A large heat flux is conducted through the low density plasma to the shell outer surface. The shell material is heated up and expands off the outer surface (the ablation process) inducing a rocket effect and accelerating the shell inward. The heat conduction and the ablation process occurring at the shell outer surface reduce the Rayleigh-Taylor growth rate to a level significantly below its classical value (see [7, 8, 9]). The instability occurring at the shell outer surface during the acceleration phase is also referred to as the "ablative" Rayleigh-Taylor instability.

The theory of the ablative RT instability has been carried out in [10, 11, 12,
The growth rates are calculated for an ablatively accelerated planar foil. The foil is treated as a fully ionized collisional plasma slab with a finite nonlinear thermal conductivity:

$$\kappa(T) = \kappa T^\nu$$  \hspace{1cm} (1.5)

where $\kappa$ and $\nu$ are constant. The ablative flow is typically subsonic with respect to both the shell and the ablated plasma sound speeds so that the pressure variations through the ablation front can be neglected (isobaric approximation, see [15]). Using this approximations, the linearized conservation equations (mass, momentum and energy) are solved using sophisticated asymptotic methods (boundary layer and WKB techniques). In spite of the complicated analytic form given in [10], the growth rate can be fitted by one of the two following formulas:

$$\gamma = \alpha \sqrt{kg - \beta kV_a}$$  \hspace{1cm} (1.6)

$$\gamma = \alpha \sqrt{\frac{kg}{1 + kL_m}} - \beta kV_a$$  \hspace{1cm} (1.7)

where $g$ is the foil acceleration and $V_a$ is the velocity at which the ablation front advances in the foil. Given the mass ablation rate off the shell $\dot{m}$ (=mass ablated per unit time per unit surface), the ablation velocity is determined by the ratio between the ablation rate and the unablated foil density

$$V_a = \frac{\dot{m}}{\rho_{\text{foil}}}$$  \hspace{1cm} (1.8)

The coefficients $\alpha$ and $\beta$ are of order unity and, as shown in [10], depend on the foil power index for thermal conduction $\nu$ and a dimensionless combination of ablation velocity, foil acceleration and density gradient scale length. Such a dimensionless parameter is known as Froude number:

$$Fr_r = \frac{V_a^2}{gL_0}$$  \hspace{1cm} (1.9)
where \( L_0 \) is a length proportional to the density gradient scale length

\[
L_0 = L_m \frac{\nu'}{(\nu + 1)^{\nu+1}}
\]  
(1.10)

The length \( L_0 \) is also related to the thermal conductivity, the ablation velocity and the ratio of specific heats \( \Gamma \):

\[
L_0 = \frac{\Gamma - 1}{\Gamma} \frac{A}{} \frac{\tilde{T}_{\text{foil}}}{T_{\text{foil}}} \frac{\rho_{\text{foil}}}{\rho_{\text{foil}}} V_a
\]  
(1.11)

where \( A = m_i / (1 + Z) \), \( m_i \) is the ion mass and \( Z \) is the atomic number. As it is shown in [10], the growth rate of accelerated foils with large Froude numbers \( (Fr > 1) \) is better fitted by a formula of the kind 1.6 while the foils with small Froude numbers \( (Fr < 1) \) obey a formula of the kind 1.7. The parameters \( \alpha \) and \( \beta \) for different values of the power index \( \nu \) and Froude numbers are plotted in the figures 5-6 of [10]. For a cryogenic DТ target, the heat transport is dominated by electronic heat conduction and the power index is \( \nu = 5/2 \). The ablation velocity is large and the Froude number is well above unity \( (Fr_{DT} \sim 4 \div 5) \). The parameters \( \alpha \) and \( \beta \) can be deduced from the figure 5 of [10] leading to the following growth rate formula:

\[
\gamma_{DT} = 0.94\sqrt{kg} - 2.7kV_a
\]  
(1.12)

The formula 1.12 is remarkably close to the one derived in [9] by Takabe et al. by means of fitting the results of the numerical solution of the linearized conservation equations \( (\gamma_{\text{Takabe}} = 0.9\sqrt{k}\xi - 3kV_a) \).

For higher-Z materials such as plastic (CH) commonly used in laser experiments, the energy transfer is dominated by radiation transport. This can also be described by an effective thermal conductivity with a power law dependence on temperature. As reported in [11], the power index \( \nu \) for plastic is near unity, the ablation velocity is small and the Froude number is typically less than unity.
Figure 1.4: The inner surface of the shell is unstable during the deceleration phase.

\( Fr \approx 0.3 \). The growth rate for plastic is better fitted by a formula of the kind 1.7 with \( \alpha = 1 \) and \( \beta = 1.7 \),

\[
\gamma = \sqrt[1 + kL_m]{\frac{k \bar{g}}{1 + kL_m}} - 1.7kV_a \quad (1.13)
\]

In conclusion, the growth rate of the ablative RT instability in an accelerated planar foil can be easily determined using the results of [10] once the ablation velocity, acceleration, density gradient scale length and power index for thermal conduction are known.

During the deceleration phase, the imploding shell is slowed down by the large pressures building up inside the hot-spot. In the shell frame of reference, the inertial force is directed from the dense shell material towards the light hot spot plasma and the shell inner surface is unstable to the Rayleigh-Taylor instability.

It is common wisdom that the deceleration phase RT instability is classical and the finite density gradient scale length is the only stabilizing mechanism. Following Lindl’s work [2], the DPRT instability growth rates are approximated by the
following heuristic formula:

$$\gamma = \sqrt{\frac{kg}{1 + kL_m}}$$  \hspace{1cm} (1.14)

where $k$ is the perturbation wavenumber. In order to satisfy the periodicity constraint, the mode wavelength must be an integer fraction of the shell circumference ($\lambda = 2\pi R/l$ where $l$ is an integer), and the mode wavenumber is $k \approx l/R$, $R$ is the hot-spot radius. According to [2], the density gradient scale length is a significant fraction of the hot-spot radius: $L \approx 0.1 \div 0.2R$. Such a large density gradient scale length leads to a significant reduction of the instability growth rates. The perturbation grows over a time $t$ during which the shell is traveling a distance $\Delta \sim gt^2/2$. Lindl argues that the shell displacement during the deceleration phase is approximately equal to the final hot spot radius $R$ thus concluding that $\Delta \sim R$ and the RT growth factor is

$$e^{\gamma t} = e^{\sqrt{\frac{2t}{1 + 0.2l}}}$$  \hspace{1cm} (1.15)

According to equation 1.15, short wavelength modes ($l \gg 1$) are unstable and their growth factor is approximately $e^{\sqrt{10}} \approx 24$. Very recently Lobatchev and Betti have shown in [17] (see also Chapter 3 of this thesis) that the deceleration phase instability is not classical, because the mass ablation off the inner shell surface significantly reduces the RT growth rates and suppresses short wavelength modes (see [12]). Mass ablation is caused by the heat flux leaving the hot-spot and depositing on the shell inner surface. We have calculated the ablation velocity and the shell density gradient scale length during the deceleration phase. Then using the ablative RT growth rates of [10], we have calculated the growth rates and compared them with the results of numerical simulations. The simulations have been carried out using the 2D Eulerian code described in Chapter 4. Contrary to the equation 1.14, it is found that for direct drive NIF-like capsules, the unstable
spectrum exhibits a cutoff at \( l \approx 90 \), i.e. all modes with \( l > 90 \) are stable.

1.3 Seeding of the Rayleigh-Taylor Instability.

The APRT instability can be seeded by both rear and front surface nonuniformities. Recent progress in laser smoothing and target manufacturing techniques has significantly reduced the laser intensity and front surface nonuniformities. However, the rear surface of cryogenic targets is usually very rough and provides the main seed for the APRT instability in both direct and indirect drive ICF.

Although only the outer shell surface is unstable during the acceleration phase, the inner surface perturbations propagate through the target and reach the outer surface where they grow exponentially. This process is typically referred to as "feedout". The feedout is induced by the initial shock wave propagating through the shell. When the laser is turned on, a shock wave travels from the outer to the inner surface leaving the compressed material in motion with a constant velocity. After the shock breaks out on the inner surface, a rarefaction wave propagates toward the outer surface. After the rarefaction front reaches the outer surface, the latter starts accelerating and becomes unstable to the RT instability. If the rear surface is rippled, the reflected rarefaction wave is also rippled. When the rippled rarefaction front reaches the outer surface, it imprints a perturbation on it which grows exponentially during the acceleration phase. It is one of the goals of this work to develop a physical understanding and quantitative prediction of the RT seeding by rear surface nonuniformities.

After the perturbation is seeded, it starts to grow exponentially as \( A e^{\gamma t} \), and the mathematical definition of the seed can be introduced as the coefficient \( A \) before the exponential term in the APRT instability growth factor.
Figure 1.5: Mode spectrum at the beginning of the acceleration phase according to the ORCHID simulations. The feedout provides the main RT seed for modes with $l < 20$.

Figure 1.5 (figure 1 of [18]) demonstrates that the feedout is the main seed of the APRT for the long-wavelength ($l < 20$) nonuniformities in NIF targets.

The negative impact on capsule performances induced by the RT instability seeded by rear surface nonuniformities have been estimated using two dimensional numerical simulations, the simulations have been performed with the LLE hydrodynamic code ORCHID (see [24]). The results are summarized in figure 1.6 where the gain of a NIF-like capsule is plotted versus the magnitude of the initial rear surface roughness. Observe the rapid fall off of the energy gain with the magnitude of the rear-surface perturbations demonstrating the detrimental effect of the feedout on target performance.

The theory of the rear-surface perturbation feedout has been developed by Betti, Lobatchev and McCrory in [25] and Chapter 2 of this thesis. The analysis has been carried out for long wavelength perturbations of planar foils accelerated by a constant laser irradiation. Long wavelength modes are characterized by the size of their wavelength (or inverse wavenumber $k$) in relation to the foil thick-
Figure 1.6: Gain of a NIF-like capsule versus the initial inner surface perturbation. 

\[ \sigma_{rms}^2 = \frac{1}{4\pi} \sum_{l,m} |R_{lm}|^2 \] is the measure of the perturbation amplitude, where \( R_{lm} \) is the amplitude of the \( l,m \) spherical harmonic.

ness \( d \). A long wavelength mode satisfies \( kd < 1 \) while a short wavelength one has \( kd > 1 \). In [25], the authors have derived a simple formula describing the outer surface distortion seeded by the inner surface nonuniformities during the acceleration phase. It is found that the largest perturbation feedout occurs for long wavelength and falls off approximately as \( 1/k \). Short wavelength modes are not included in the theory, and they are studied numerically. A two dimensional Lagrangian code has been developed to simulate both short and long wavelength feedout. The details of the code are given in Chapter 4. As described in Chapter 2, simulations of the feedout process for various mode wavelengths and material properties has been used to modify the analytic formula derived for \( kd < 1 \) and to extend its validity to the short wavelength regime (\( kd > 1 \)).
Chapter 2

Seeding of the Acceleration Phase Rayleigh-Taylor Instability by Inner Surface Nonuniformities.

2.1 Introduction.

The acceleration phase Rayleigh-Taylor instability (APRT) of an imploding shell is seeded by nonuniformities of the laser beam intensity (laser imprinting) as well as nonuniformities of the shell surfaces. Recent advances in laser smoothing techniques have greatly improved the beam nonuniformities. According to the stability analysis and 2D simulations of NIF-like targets conducted at the Laboratory for Laser Energetics (see [18]), the current generation of two dimensional 1 THz SSD system (Smoothing by Spectral Dispersion, see [26]) combined with DPPs (Distributed Phase Plates, see [27]) and DPRs (Distributed Polarization Rotators) should be able to guarantee the integrity of the shell during the acceleration phase.
It is likely that the greatest threat to the shell integrity comes from the inner surface nonuniformities. The inner surface roughness is typically large and difficult to control during the shell manufacturing process. During the initial phase of the shell implosion, the inner surface perturbations travel toward the outer surface with the rarefaction wave originating at the inner surface. Once the perturbations have reached the outer surface, they are amplified by the RT instability and the shell outer surface distortion amplitude \( \Delta_{out} \) grows exponentially in time:

\[
\Delta_{out} = Ae^{\gamma t}
\]

where \( \gamma \) is the RT growth rate and \( A \) is a constant.

The convergence effects can be neglected in the analysis because all the unstable modes have wavenumbers \( l > 1 \), and the feedout occurs during the initial stage.
of the implosion, when the shell aspect ratio (shell radius/shell thickness) is large. A constant irradiation is typical for the initial phase of ICF implosions when the laser intensity is constant and a uniform shock is propagating through the target. It is during this initial stage that the RT seeding occurs. Thus, it is meaningful to investigate the feedout process in planar geometry and to consider the RT seeding in a planar foil accelerated by a constant laser irradiation. In planar geometry there is not an inner or outer surface. The laser irradiated surface of the foil plays the role of the outer shell surface while the rear foil surface simulate the shell inner surface. In the rest of this chapter we refer to the laser irradiated surface of the foil as the front surface and to the opposite side as the rear surface.

In Section 2.2 of this chapter, we introduce the ideal gas hydrodynamic equations which constitute the basic model used to describe the foil evolution. In Section 2.3, we determine the one dimensional motion of a planar foil irradiated by a constant laser intensity. In Section 2.4, we study the perturbation transfer from the rear to the front surface and subsequent exponential growth of perturbations with wavelengths longer than the target thickness \((kd_{ps} < 1 \text{ where } k = 2\pi/\lambda\) is the mode wavenumber and \(d_{ps}\) is the post-shock foil thickness). The analysis is carried out analytically by solving the linearized ideal gas hydrodynamic equations. The final product of the analytic theory is a formula relating the constant \(A\) in the equation 2.1 with the initial rear surface perturbation, mode wavenumber and foil thickness. In Section 2.5, we describe the results of numerical simulations performed using the 2D Lagrangian code and compare the simulation results with the analytic theory. In Section 2.6, we use the results of numerous 2D simulations of short wavelength feedout \((kd_{ps} > 1)\) to construct a correcting factor to the long wavelength analytic formula and extend its validity into the short wavelength regime.
2.2 Hydrodynamic Equations of Motion.

After the shock propagation, the foil is usually fully ionized, its temperature is typically in the few electronvolt range and its density is a few grams per cubic centimeter.

At such temperatures and densities, the electron-ion collision time scale is much shorter than the typical time of any hydrodynamic phenomenon of interest. Furthermore, the energy equilibration is so fast that ion and electron temperatures are approximately equal to each other. It is well known that the dynamics of such a collisional plasma can be described by the single fluid hydrodynamic equations shown below:

1. Conservation of mass

\[
\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \bar{u}) = 0 \tag{2.2}
\]

2. Conservation of momentum

\[
\frac{\partial}{\partial t} \bar{u} + (\bar{u} \cdot \nabla) \bar{u} = -\frac{1}{\rho} \nabla P \tag{2.3}
\]

3. Conservation of energy

\[
\rho \left( \frac{\partial}{\partial t} + \bar{u} \cdot \nabla \right) E = -P \nabla \cdot \bar{u} - \nabla \cdot \bar{q} + S \tag{2.4}
\]

4. Ideal gas equation of state

\[
E = E(P, \rho) = \frac{P}{\rho(\Gamma - 1)} \tag{2.5}
\]

5. Fourier/Spitzer law for heat conduction

\[
\bar{q} = -\kappa(T) \nabla T \tag{2.6}
\]

where \( \rho \) is the density, \( \bar{u} \) is the fluid velocity, \( P \) is the pressure, \( \Gamma = c_p/c_v \) is the ratio of specific heats under constant pressure and volume, \( E \) is the specific
energy, \( \bar{q} \) and \( S \) are the heat flux and the heat source term respectively. The target material is modeled as an ideal gas with pressure, density and energy related by the ideal gas equation of state. The temperature is proportional to the specific energy: 

\[
T = EA(T - 1) \quad \text{where} \quad A = \frac{m_i}{(1 + Z)}, \quad m_i \text{ is the ion mass and } Z \text{ is the atomic number. The heat flux is proportional to the temperature gradient and the heat conductivity has a power law dependence on temperature (see [28, 29]).}
\]

\[
\kappa(T) = \kappa T^\nu
\]

(2.7)

where \( \nu \) is the power index for thermal conduction and \( \kappa \) is approximately constant. For low-Z materials such as hydrogen isotopes, the energy transfer is dominated by electronic heat conduction described by the conductivity 2.7 with \( \nu = 5/2 \) and the constant \( \kappa \) (in CGS units) given by

\[
\kappa = \frac{2.4}{\sqrt{2\pi}} \frac{1}{\sqrt{m_e e^4 \Lambda}}
\]

(2.8)

where \( m_e \) and \( e \) are the electron mass and charge respectively, \( \Lambda \) is the Coulomb logarithm, and the temperature \( T \) in 2.7 is measured in ergs (see [30]).

If \( Z > 1 \), the radiation energy transfer becomes important. The radiation (both bremsstrahlung and line radiation) is emitted from the corona and absorbed near the ablation front. Higher frequency radiation is absorbed deeper in the target and the standard models for radiation transport require the solution of multigroup diffusion equations (see [31]), with the radiation spectrum divided into several energy groups each governed by a diffusion equation. Multigroup radiation transport has been implemented in the main stream ICF codes such as LILAC (1D, LLE), ORCHID (2D, LLE), LASNEX (1 and 2D, LLNL), FAST2D (NRL), FCI2 (2D, CEA). From the numerical viewpoint, the radiation diffusion operator requires an inversion of ill-conditioned matrix which is very expensive in terms
of computer time. Typically 90% of the computer time needed to run any of these codes is spent for the multigroup radiation transport solver. Furthermore, it is virtually impossible to carry out any detailed analytic theory of the coupled system of plasma-radiation conservation equations.

For such reasons, we include the radiation transport by adopting a Zeldovich-like approach (see [32]) for a plasma in local thermodynamic equilibrium with the radiation field and replace the Spitzer conductivity with an ad hoc power law. In other words, the energy flux $\bar{q}$ is still described by the Fourier law $[\bar{q} = -\kappa \nabla T]$ where $T$ represents both the plasma and radiation temperature. Similarly to the electronic heat conduction case, the conductivity is still proportional to $T^{\nu}$, however the power index $\nu$ is determined by a best fit of the one dimensional multigroup results. This approximation has been successfully used in [11] to study the growth of the ablative R-T instability. For the power law dependence of the thermal conductivity, the steady state ablation front profiles can be calculated analytically. The power index $\nu$ and the constant $\bar{\kappa}$ are determined by fitting such analytic profiles with the profiles obtained by one dimensional simulations carried out using the multigroup radiation transport code LILAC. It is found that most ICF relevant high-Z materials (such as plastic and beryllium) are well described by $\nu \approx 1$.

In conclusion, whether or not the energy transfer is dominated by radiation transport over electronic heat conduction, we make use of the equation 2.2-2.6 with the power law temperature dependence of the heat conductivity. The power index $\nu$ is equal to 5/2 for low-Z target materials (such as DT) and $\nu = 1$ for higher Z materials such as plastic and beryllium. The values of $\bar{\kappa}$ for the different materials can be deduced from the values of the parameter $L_0$ tabulated in [10].

The laser energy absorption is approximated by a heat source term $S$ localized
at the critical density surface where the plasma frequency is equal to the laser frequency. In mathematical terms, the source is approximated by the function $S = I_L \delta(\vec{x} - \vec{x}_{cr})$ where $I_L$ is the absorbed laser intensity and $\vec{x}_{cr}$ represents the location of the critical surface. In the numerical code the source term is distributed among several grid points.

Equations 2.2-2.6 represent the complete set of hydrodynamic equations describing the motion of a laser accelerated foil. The next step is to carry out the solution of the one dimensional hydro equations for the foil. The solution is developed both analytically and numerically using standard techniques described below.

### 2.3 One Dimensional Evolution of an Accelerated Planar Foil.

As the foil is irradiated by the laser light, its material is ablated off the front surface of the target and a strong ablation pressure builds up. As a consequence of the sudden rise in pressure, an ionizing shock wave propagates through the foil. After the shock breaks out on the rear surface, the gas expands isentropically and a reflected rarefaction wave propagates toward the front surface. After the rarefaction wave breaks out on the front (ablation) surface, the foil starts accelerating and the front surface becomes RT unstable.

Three different regions can be identified after the beginning of the laser irradiation. The *corona* is the region where the plasma with densities below the critical density expands isothermally and the temperatures are of the order of a few keVs. The *blow-off region* is the region between the critical surface and the ablation sur-
Figure 2.2: One-dimensional density profiles during the shock and rarefaction wave propagation. Due to the high ablation pressure the shock wave compresses the target. The reflected expansion then propagates toward the front surface, and after its break-out the front surface acceleration begins.

Surface where the plasma is heated up by the heat flux conducted from the critical surface. The temperature in this region changes dramatically from the cold foil temperature of a few electronvolts to the coronal temperatures of a few keVs. The overdense foil region is the cold dense foil plasma ionized and heated by the initial shock up to temperatures of a few eVs.

During the initial stage of the implosion when the RT seeding occur, the ablated mass is a small fraction of the total foil mass. The ablation pressure in the blow-off region is approximately constant because the laser intensity is constant. Following [2], the ablation pressure can be related to the laser intensity by the following simple formula

\[ P_a = 40 \left( \frac{I_{15}}{\lambda} \right)^{2/3}, \]  

(2.9)

where the pressure \( P_a \) is measured in \( Mbars \), the laser intensity \( I_{15} \) is in units of \( 10^{15} W/cm^2 \), laser wavelength is in micrometers. Due to the approximately constant mass, the foil behaves as a slab of compressible material subject to a
constant pressure given by 2.9. Thermal conduction, radiation transport and the laser absorption determine the hydrodynamics of the corona and blow-off region but their only impact on the overdense foil is via the ablation pressure. Thus, it is meaningful to model a laser accelerated foil with a compressible slab driven by a constant pressure. Such a simple model breaks down when the ablated mass becomes a significant fraction of the total foil mass. This occurs later in time after the RT seeding has taken place and the laser power has raised several times ($\geq 50$) from its initial level.

Thus, the essential physics relevant to the initial phase of the implosion and the RT seeding can be described by the simple compressible slab model. In this section we determine the one-dimensional evolution of a planar foil accelerated by a large constant pressure by solving equations 2.2-2.5 and neglecting the heat source and thermal conduction terms in the energy equation.

The planar foil has a thickness $d_0$ and density $\rho_0$. At time $t = 0^-$, the outside pressure is $p_0$. At time $t = 0^+$, the pressure $p_a \gg p_0$ is applied on one side of the foil via laser illumination at $x = d_0$ while the pressure on the other side is kept constant at $p_0$.

In typical experimental setups, the planar foil is originally solid and located in a vacuum chamber. Before the ionization occurs, we treat the solid foil as an ideal gas with a pressure $p_f$ and the surroundings as a perfect vacuum ($p_0 = 0$). The initial foil pressure ($p_f$) only affects the shock speed and can be regarded as an adjustable parameter. As a consequence of the sudden rise in pressure, an ionizing shock wave propagates through the foil. The shock velocity $U_s$ depends on the shock strength $p_a/p_f$ and the sound speed ahead of the shock $a_f$. The post-shock sound speed $a_{ps}$, flow velocity $U_{ps}$ and the gas density $\rho_{ps}$ can also be expressed in terms of the shock strength and the physical properties ahead of the shock using
the Hugoniot relations (see [33]). As a result of the shock propagation, the target is ionized, compressed and its thickness is \( \frac{d_{ps}}{\rho_{0}d_{0}/\rho_{ps}} \). The compressed target material is treated as an ideal gas with a ratio of specific heats \( \Gamma = 5/3 \). After the shock breaks-out on the back surface at time \( t_{sb} = d_{0}/U_{s} \), the gas expands and a rarefaction front propagates towards the front surface. After the shock breaks out, the flow is isentropic and \( p = p_{ps}(\rho/\rho_{ps})^{\Gamma} \). The pressure and density on the expanding back surface are fixed \( p_{b} \approx 0 \) and \( \rho_{b} \approx 0 \). The rarefaction front travels with the post-shock sound speed and reaches the front surface at the rarefaction front break-out time \( t_{rb} = d_{ps}/a_{ps} \). The rarefaction wave solution is well known and it can be easily written in the Lagrangian frame of reference defined by the initial position \( \bar{x} \) of the fluid elements at the time of the shock break-out. The Lagrangian transformation \( (x, t) \rightarrow (\bar{x}, \tau) \) can be expressed through the fluid element trajectories,

\[
x = \bar{x} + \int_{0}^{\tau} U[\bar{x}, \tau'] d\tau'
\]

where \( \tau = t - t_{sb} \) and \( 0 < \bar{x} < d_{ps} \). Solving the mass, momentum and entropy conservation equations and using the dimensionless coordinate \( \xi = \bar{x}/(a_{ps}\tau) \), leads to the following form of the rarefaction wave solution:

\[
\rho = \rho_{ps}\xi^{2\Gamma+1}, \quad p = p_{ps}\xi^{2\Gamma+1}, \quad U = \frac{2a_{ps}}{\Gamma - 1} (\xi^{\Gamma+1} - 1) + U_{ps}, \quad (2.11)
\]

where \( 0 < \frac{\bar{x}}{\tau} < a_{ps} \);

\[
\rho = \rho_{ps}, \quad p = p_{ps}, \quad U = U_{ps}, \quad (2.12)
\]

where \( a_{ps}\tau < \bar{x} < d_{ps} \)

where \( U \) and \( U_{ps} \) are both negative because the shock-induced motion is directed in the negative \( x \) direction. In the target frame of reference, 2.11-2.12 are equivalent to the rarefaction wave solution found in [34]. Equation 2.11 shows that
the rarefaction fan boundaries move in the opposite directions in the target frame of reference. The target-vacuum interface denoted as an expansion front travels with the escape velocity \( U_{es} = U_{ps} - 2a_{ps}/(\Gamma - 1) \) and the rarefaction front with the velocity \( U_r = a_{ps} + U_{ps} \). The rarefaction wave solution is only valid until the rarefaction wave reaches the front surface.

Since the applied pressure is constant, the front surface moves at constant velocity until the rarefaction wave breaks out. At \( t = t_{rb} \), the front surface starts accelerating. As shown later, it is crucial for the feedout imprinting problem and early R-T growth to determine the acceleration at time \( t \geq t_{rb} \). A similar problem is solved in [35] for the case of a rarefaction wave incident on a fixed surface rather than a free surface. When the rarefaction wave interacts with a free surface where the pressure is assigned, the problem can be greatly simplified by using the hodograph method in the Lagrangian frame of reference \((\bar{x}, \tau)\). The Riemann invariants \( J^+ = 2a/(\Gamma - 1) + u, J^- = 2a/(\Gamma - 1) - u \) are constant along the characteristic curves

\[
C^+ : \quad \frac{d\bar{x}}{d\tau} = \hat{a}, \quad C^- : \quad \frac{d\bar{x}}{d\tau} = -\hat{a} \tag{2.13}
\]

respectively, where \( \hat{a} = a\rho/\rho_{ps} \) and \( a = \sqrt{\Gamma p/\rho} \) is the sound speed. Using the Riemann invariants as new coordinates replacing \((\bar{x}, \tau)\), the characteristic equations can be combined into a single PDE:

\[
\frac{2(\Gamma - 1)}{\Gamma + 1} \frac{\partial^2 \tau}{\partial J^+ \partial J^-} + \frac{1}{J^+ + J^-} \left[ \frac{\partial \tau}{\partial J^+} + \frac{\partial \tau}{\partial J^-} \right] = 0 \tag{2.14}
\]

Equation 2.14 is identical to the equation studied by D’Alambert with the exact solution

\[
\tau = \frac{\partial}{\partial J^+} \left[ \frac{f(J^+)}{(J^+ + J^-)^2} \right] + \frac{\partial}{\partial J^-} \left[ \frac{f(J^-)}{(J^+ + J^-)^2} \right] \tag{2.15}
\]
for $\Gamma = 5/3$, $f$ and $g$ are two arbitrary functions here. Once the equation 2.14 is solved, the variable $\bar{x}$ can be found using the characteristic equation $d\bar{x} = -\dot{\bar{x}}d\tau$ for $J^- = \text{const}$, yielding

$$\frac{\partial \bar{x}}{\partial J^+} = -\dot{\bar{x}} \frac{\partial \tau}{\partial J^+} \quad (2.16)$$

The solution of the equation 2.14 gives the time $\tau$ as a function of the Riemann invariants: $\tau = \tau(J^+, J^-)$. The equation 2.14 needs to be solved in the domain (region III) shown in figure 2.3, between the characteristic $C_0^-$ and the front surface $\bar{x} = d_{ps}$. In region II, $\dot{\bar{x}} = \bar{x}/\tau$ and $C_0^-$ is given by the equation $d\bar{x}/d\tau = -\bar{x}/\tau$ with initial condition $\bar{x} = d_{ps}$ for $\tau = d_{ps}/a_{ps}$. Thus, the curve $C_0^-$ is given by $\bar{x} = d_{ps}^2/a_{ps}\tau$ and the Riemann invariants on $C_0^-$ can be written in the following form:

$$J^- = J_0^- = \frac{2a_{ps}}{\Gamma - 1} - U_{ps}, \quad J^+ = \frac{2a_{ps}}{\Gamma - 1} \left[ 2 \left( \frac{d}{a_{ps}\tau} \right)^{2^{\Gamma+1}} - 1 \right] \quad (2.17)$$
The last equation can be inverted yielding $\tau = \tau(J^+)$ for $J^- = J_0^-$. The other side of the region III is the front surface at $\bar{x} = d_{ps}$ where the pressure (and therefore the density and sound speed) is assigned and $J^+ + J^- = 4a_{ps}/(\Gamma - 1)$ is constant. Since $d\bar{x} = 0$ and $dT^+ = -dT^-$ on the front surface, using the characteristic equations leads to the following form of the boundary condition on the front surface

$$\frac{\partial \tau}{\partial J^+} + \frac{\partial \tau}{\partial J^-} = 0 \quad \text{for } \bar{x} = d_{ps} \quad (2.18)$$

The solution of 2.14 satisfying the boundary conditions 2.17,2.18 can be found for $\Gamma = 5/3$ and written in the following form

$$T = \frac{e^{\sqrt{3}J}}{A^3} \left[ \hat{A}(\cos \hat{J} + \sqrt{3} \sin \hat{J}) - \frac{2}{\sqrt{3}} \sin \hat{J} \right] \quad (2.19)$$

where $T = a_{ps} \tau / d_{ps}$, $\hat{A} = a / a_{ps}$ and $\hat{J} = (J^- - 3a_{ps})/(2\sqrt{3}a_{ps})$. The velocity of the front surface $u_{fs}$ can be obtained by setting $\hat{A} = 1$ and $\hat{J} = -u_{fs}/(2\sqrt{3}a_{ps})$ into 2.19. A straightforward manipulation leads to the following implicit equation for the front surface velocity and acceleration

$$T = \frac{2}{\sqrt{3}} \cos \left( \frac{u_{fs}(T)}{2\sqrt{3}a_{ps}} + \frac{\pi}{6} \right) \exp \left( -\frac{u_{fs}(T)}{2a_{ps}} \right) \quad (2.20)$$

$$g_{fs}(T) = \frac{a_{ps}}{d_{ps}} \frac{du_{fs}}{dT} \quad (2.21)$$

The analytic form of the acceleration 2.21 has been compared with the result of 1D numerical Lagrangian simulation. We have carried out numerical simulations of a planar foil of thickness $d_0 = 20\mu m$ and density $\rho = 1gr/cm^3$ accelerated by a 20Mbar pressure using the code described in Section 4.1. The front surface acceleration derived from the numerical simulation and the analytical result 2.21 are compared in figure 2.4. Observe that the analytical and numerical results
Figure 2.4: Evolution of the front surface acceleration. The dashed line is the analytic prediction 2.21, the solid line is the numerical result. Theory and simulation agree as long as the flow is isentropic. The steady state value 2.23 is reached after $\sim 1.2\,\text{ns}$. 
agree quite well during a time interval of approximately 0.5 ns after the rarefaction wave breaks down. Later in time the analytical result deviates considerably from the numerical simulation. The discrepancy can be explained by the fact that the acceleration given by 2.21 is only valid as long as the flow is isentropic. It is easy to show that the Jacobian of the hodograph transformation vanishes when \( \tau = (9/4)(d_{ps}/a_{ps}) \) at \( \bar{x} = (4/9)d_{ps} \) indicating that a secondary shock forms inside the target during the acceleration phase. Thus, one can conclude that the equations 2.20-2.21 are valid until \( \tau = (9/4)(d_{ps}/a_{ps}) \) (or sometime after that) when the isentropic flow assumption breaks down. Figure 2.5 shows the density profile

![1-D distribution of density inside the target](image)

**Figure 2.5:** Density profile obtained in numerical simulations shortly after the secondary shock is formed. The isentropic theory is no longer applicable for \( t - t_{rb} > 0.42 \text{ ns} \).

at the time of the secondary shock formation. The density profile is the result of a numerical simulation of the target mentioned above. The steepening of the density
profile occurring inside the foil is a clear indication of the secondary shock formation. Once formed, the shock travels down the density profile, gaining strength and setting the rear surface of the foil on a very high adiabat. The secondary shock forms at 0.8\(ns\) and the isentropic approximation breaks down sometime after that at about 1.2\(ns\) as shown in figure 2.5.

At the rarefaction wave break-out time, the equations 2.20-2.21 yield the front surface acceleration

\[ g_{fs}(t_{rb}) = \frac{5p_{a}}{(2\rho_{0}d_{0})} \]  

(2.22)

At later times, the front surface acceleration decreases and reaches the quasi steady state value:

\[ g_{fs} \to \frac{p_{a}}{\rho_{0}d_{0}} \text{ as } t \to \infty. \]  

(2.23)

It is therefore appropriate to represent the front surface acceleration using 2.21 up to about 1.2\(ns\) and to use the steady state value after that. It is important to observe that because of the sudden rise of the acceleration, the simulation does not accurately reproduces the acceleration peak at the rarefaction wave break-out time \(t_{rb}\) unless an extremely fine grid is used. Thus, a great care must be taken if the simulated acceleration (instead of the exact analytic calculation) is used starting from the rarefaction wave break-out time. This concludes the analysis of the 1-D motion of the planar foil subject to a constant pressure \(p_{a}\). The next step is to determine the two dimensional evolution of the foil in the presence of a rippled back surface.
2.4 Feedout of Long Wavelength Modes.

If the back surface is rippled, the shock-wave rippled-surface interaction produces a rippled rarefaction front (see [36, 37, 38]). As the rippled rarefaction front breaks out on the front surface, a perturbation is imprinted on it and grows exponentially in time during the acceleration phase. The exponential growth does not develop immediately after the rarefaction wave break out. If a velocity and/or acceleration perturbations are imprinted on the front surface by the rippled rarefaction wave, the distortion first grows linearly in time due to the velocity perturbation, then quadratically due to the acceleration perturbation. It is only after $\Delta t \sim \gamma^{-1}$, that the exponential growth is established. Since $\gamma \sim \sqrt{k g}$, long wavelength modes develop the exponential growth later than short wavelengths. Thus for $t > \Delta t$ the front surface distortion amplitude during the acceleration phase grows exponentially and it can be written as

$$\Delta f_s(t > \Delta t) \simeq \Delta_0 F \exp \left\{ \int_{t_{rb}}^t \gamma dt' \right\}$$

(2.24)

where $\gamma$ is the RT growth rate, $t_{rb}$ is the rarefaction wave break out time, $\Delta_0$ is the initial rear surface perturbation and $F$ is the so called transfer function. If $F = 1$, then 100% of the rear surface perturbation is transferred to the front surface. If $F > 1$ then the perturbation has been amplified by the feedout. If $F < 1$ the perturbation has been reduced. It is the goal of this section to determine the transfer function $F$ for modes with the inverse wavenumber greater than the post-shock target thickness ($kd_{ps} < 1$), i.e. long wavelength modes.

We consider an initial rear surface ripple $\Delta_0 \cos(ky + \pi)$. Shortly after the shock breaks out on the back surface, the ripple on the rarefaction front can be determined using the following considerations. For long wavelength modes, the motion parallel to the shock front can be neglected and the rarefaction occurs
perpendicularly to the shock front. As shown in the figure 2.6, the shock first reaches the peaks of the ripple where the leading and the trailing edges of the rarefaction originate first.

Two-dimensional behavior in the presence of a rippled rear surface

(a) The shock reaches the rippled rear surface.

(b) Shock and rippled surface interact.

(c) The rippled rarefaction front is formed.

(d) The rarefaction reaches the front surface.

Figure 2.6: Two-dimensional qualitative behavior in the presence of a rippled rear surface. (a)-(c) show the formation of the rippled expansion wave, the leading edge perturbation is approximately 80% of the initial back surface distortion \( \Delta_0 \); (d) corresponds to the interaction between the front surface and the rarefaction wave and the beginning of the perturbation seeding process at the front surface. The scale is not kept constant for all pictures.

The rarefaction fan widens while the shock travels towards the ripple valleys (figure 2.6). As shown earlier, the velocity of the rarefaction front is \( U_r \). By the
Figure 2.7: When the rippled rarefaction wave reaches the front surface, the compression wave forms and starts propagating toward the back surface. A perturbation is seeded on the front surface and begins to grow. The picture shows the density isolines shortly after the rarefaction break-out.

time the shock is reaching the ripple valleys, the leading edge originated at the peaks has moved by $U_r \Delta_0 / U_s$. Thus, the ripples on the rarefaction front right after the shock has reached the valleys is $\Delta_r \cos(ky + \pi) = \Delta_0 (1 + U_r/U_s) \cos(ky + \pi)$. The latter represents the initial conditions for the rarefaction wave propagation. Since we neglect the transversal (along $y$) flow (proportional to $\epsilon \equiv kd_{ps} < 1$), the motion of peaks and valleys is essentially 1-D until the rarefaction wave breaks out on the front surface. That is, in the frame of reference of the front surface, the peak and valley of the rarefaction front travel at the sound speed $a_{ps}$ toward the front surface thus keeping the ripple amplitude constant (to lowest order in $kd_{ps}$).

Once the rippled leading edge of the rarefaction approaches the front surface,
Figure 2.8: The first figure shows the interaction between the front surface and the peak of the rippled rarefaction front. The second figure shows the rarefaction wave when the ripple valley has reached the front surface. The time interval between them is $\Delta t$.

The ripple peaks experience the acceleration $g_{fs}$ before the valleys (see figure 2.8). The valleys reach the front surface with a delay $\Delta t = \Delta r / a_{ps}$. During that time the peaks have been accelerated to the velocity $\delta v_x = g_{fs}(t_{rb}) \Delta t$. It follows that by the time the entire rarefaction front (peak and valley) has reached the front surface, a velocity perturbation is imprinted on the latter. The resulting velocity perturbation $\delta v_x \cos(ky)$ imprinted at the rarefaction wave break-out time on the front surface by the feedout process is

$$\delta v_x(t_{rb}) = g_{fs}(t_{rb}) \frac{\Delta r}{a_{ps}},$$  \hspace{1cm} (2.25)

where $g_{fs}(t_{rb})$ is given by 2.22. Furthermore, the target mass under the peak of the
initial perturbation is less than under the valley because the target thickness under the peak is \( \Delta_0 \) less than under the valleys. Since the target acceleration depends on the mass/thickness, a perturbation in the acceleration \( \delta g_{fs} \cos(ky) \) develops between the peaks and the valley:

\[
\delta g_{fs} = -\frac{\partial g_{fs}}{\partial \Delta_0} \Delta_0
\]  

(2.26)

At later times \((t > t_r)\), the acceleration reaches a quasi-steady state \( g_{fs} \rightarrow p_a/(\rho_0 d_0) \) and the perturbed acceleration becomes

\[
\delta g_{fs} \rightarrow \frac{g_{fs} \Delta_0}{d_0}
\]  

(2.27)

where \( g_{fs} \) is given by 2.21 as long as the flow is isentropic. Figure 2.9 shows the

![Diagram showing qualitative behavior and time evolution](image)

Figure 2.9: Top figure shows the qualitative behavior of an accelerated foil with a rear surface perturbation. The bottom figure shows the time evolution of the perturbed acceleration.
time evolution of the perturbed acceleration calculated using the equations 2.26 and 2.27 for the foil described earlier and an initial rear surface perturbation of 0.05$\mu$m. The dashed curve on the right side represents the analytic results obtained by using the equations 2.20, 2.21 and 2.26. The solid line represents the numerical evaluation of 2.26 and the dotted line indicates the asymptotic value as predicted by 2.27. The analytic result is exact as long as the flow is isentropic. Since a secondary shock starts forming at 0.8 ns, we expect a break down of the analytic theory shortly after 0.8 ns (as shown in figure 2.9).

We conclude that the feedout process of long wavelength modes leads to the imprinting of a velocity and an acceleration perturbation on the front surface. Equations 2.25, 2.26 are derived as a lowest order expansion in $kd_{ps} < 1$ because any transversal motion has been neglected. It is important to observe that no surface perturbations are imprinted on the front surface and the velocity and acceleration perturbations are independent of the wavenumber and a secular distortion (not exponential) occurs even in the limit $k \to 0$ corresponding to a stable R-T mode.

After the rarefaction wave break-out, the imprinted velocity and acceleration perturbations are seeded and the Rayleigh-Taylor instability starts. In the limit of $k \to 0$, no R-T growth occurs and the front surface distortion $\Delta_{fs}$ grows secularly because of the imprinted velocity and acceleration perturbation described by the following ODE and initial conditions:

$$\left[ \frac{d^2 \Delta_{fs}}{dt^2} \right]_{k \to 0} = \delta g_{fs}, \quad \Delta_{fs}(t_{rb}) = 0, \quad \Delta'_{fs}(t_{rb}) = \delta v_x(t_{rb}) \quad (2.28)$$

Initially, the ripple is driven by the imprinted velocity perturbation and grows linearly in time as $t - t_{rb}$. The perturbed acceleration then modifies the linear growth. Observe that right after the rarefaction wave break out time (see figure 2.9), the perturbed acceleration is negative thus indicating that a linear growth
should slow down. Later in time, the perturbed acceleration reaches its positive steady state value leading to a quadratic secular distortion growing like \((t - t_{rb})^2\).

For finite wave numbers, the RT instability takes over later in time and the rippled amplitude grows exponentially. Unfortunately, the exact 1-D and, therefore, linearized 2-D compressible fluid equations cannot be analytically solved after the rarefaction wave break out time. Thus, we must rely on a simplified model and physical intuition to study the RT growth from the imprinted perturbations. Such a model is generated by assuming that the RT is a surface instability inducing an incompressible perturbed velocity profile and by neglecting the effect of ablation on long wavelength modes.

A careful comparison with numerical simulations will then determine the validity of the R-T model.

Let us consider an incompressible layer of thickness \(d_{ps}\) (equal to the thickness of the compressed foil) subject to the acceleration \(g(t)\) in vacuum. A very simple linear analysis shows that the front \(\Delta_{fs} \cos(ky)\) and rear surface \(\Delta_{rs} \cos(ky)\) distortions produced by long wavelength modes can be written as

\[
\Delta_{fs} = (\Delta_1 + \Delta_2), \quad \Delta_{rs} = [\Delta_1(1 - kd_0) + \Delta_2(1 + kd_0)],
\]

(2.29)

where \(\Delta_1\) and \(\Delta_2\) represent a couple of growing/damped and oscillatory modes satisfying the following ODEs:

\[
\ddot{\Delta}_1 = kg(t)\Delta_1, \quad \ddot{\Delta}_2 = -kg(t)\Delta_2
\]

(2.30)

Four initial conditions are needed to solve the two ODEs. For the feedout problem, two initial conditions are provided by the front surface and velocity perturbation at the rarefaction wave break-out time: \(\Delta_{fs}(t_{rb}) = 0\) and \(\Delta'_{fs}(t_{rb}) = \delta v_x(t_{rb})\). Furthermore, since the flow is incompressible, \(\Delta'_{rs}(t_{rb}) = \Delta'_{fs}(t_{rb})\). The last initial
condition is provided by assigning $\Delta_{rs}(t_{rb})$ in such a way that the asymptotic value of the acceleration perturbation is the same as in the real compressible problem $\delta g_{inc} = g(t)\Delta_0/d_0$ (see the equation 2.27 above). However, during a brief transient after the rarefaction wave breaks out, $\delta g \neq \delta g_{inc}$. In order to correctly include the secular distortion occurring during this transient, we add a non homogeneous term into the equations for $\Delta_1$ and $\Delta_2$ reproducing the exact $\delta g$ and secular distortion for $k \to 0$. The final model describing the secular distortion as well as the R-T growth can be written in the following final form

$$
\ddot{\Delta}_1 = k g_{fs}(t) \Delta_1 + \frac{1}{2}(\delta g_{fs} - g_{fs}(t)\Delta_0/d_0) , 
$$

$$
\ddot{\Delta}_2 = -k g_{fs}(t) \Delta_2 + \frac{1}{2}(\delta g_{fs} - g_{fs}(t)\Delta_0/d_0) , 
$$

where $g_{fs}$ is the front surface acceleration and $\delta g_{fs} = (\partial g_{fs}/\partial d_0)\Delta_0$ is the exact perturbed acceleration. The front surface distortion is given by $\Delta_{fs} = \Delta_1 + \Delta_2$. The equations 2.31-2.32 must be solved using the following initial conditions:

$$
\Delta_1(t_{rb}) = -\Delta_2(t_{rb}) = \Delta_0/(2kd_0)
$$

$$
\Delta_1'(t_{rb}) = \Delta_2'(t_{rb}) = \delta v_x(t_{rb})/2
$$

where $\Delta_0$ and $d_0$ are the initial rear surface perturbation amplitude and target thickness respectively and $\delta v_x(t_{rb})$ is the imprinted velocity perturbation as given by the equation 2.25. It is easy to show that the equations 2.31-2.34 are reduced to 2.28 for $k \to 0$ thus yielding the correct secular distortion for long wavelength modes.

Equations 2.31-2.34 can be exactly solved when the time variation of the acceleration is neglected ($\dot{g}/g < \sqrt{k}\dot{g}$). In this case, the time asymptotic behavior of the solution for $t > t_{rb} + 1/\sqrt{k}\dot{g}$ can be approximated by the following simple
formula
\[ \Delta_{f_s} \approx \frac{1}{4} \left( \frac{\Delta_0}{kd_0} + \sqrt{\frac{3}{5}} \frac{\rho_{ps}}{\rho_0} \frac{\Delta_r}{\sqrt{kd_0}} \right) \exp \left[ \int_{t_{rb}}^{t} \sqrt{k g_{fs}(t') dt'} \right] \]  
(2.35)

For strong shocks \( \Delta_r \approx \Delta_0 \left( 1 + \sqrt{5} \right)/4 \approx 0.8 \Delta_0 \) and \( \rho_{ps} \approx 4 \rho_0 \), and the solution 2.35 can be rewritten in terms of a transfer function:
\[ \Delta_{f_s} = \Delta_0 F(kd_{ps}) \exp \left[ \int_{t_{rb}}^{t} \sqrt{k g_{fs}(t') dt'} \right] \]  
(2.36)

where \( F \) is the transfer function:
\[ F = \frac{1}{16} \left( \frac{1}{kd_{ps}} + \frac{2.5}{\sqrt{kd_{ps}}} \right) \]  
(2.37)

Equation 2.36 represents an approximate, yet accurate, formula for the front surface distortion seeded by a rear surface perturbation amplitude \( \Delta_0 \). Observe that the RT induced distortion depends on the wavenumber through the growth rate as well as the coefficient of the exponential. The latter increases as \( k \) decreases.

It is easy to show that the first and second terms in the coefficient are produced by the imprinted acceleration and velocity perturbations respectively. It is also important to emphasize that long wavelength modes start develop the exponential growth later in time with respect to short wavelength ones. Thus, the formula 2.36 has to be applied judiciously by making sure that \( t > \Delta t = 1/\gamma \).

### 2.5 Simulation of Long Wavelength Feedout.

The accuracy of formula 2.36 and the model represented by 2.31-2.34 has been tested by comparing the analytic results with two-dimensional Lagrangian simulations of an ideal gas layer accelerated by an applied constant pressure. The simulations have been carried out by using the two-dimensional Lagrangian code described in section 4.1.
We consider a foil with the initial density \( \rho_0 = 1g/cm^3 \), initial pressure \( p_f = 10Kbar \) and the initial rear surface perturbation \( \Delta_0 = 0.05\mu m \). At time \( t = 0 \), the pressure \( p_a = 20Mbar \) is applied onto the front surface. The shock and rarefaction wave break-out times are \( t_{sb} = 0.4\eta s \) and \( t_{rb} = 0.58\eta s \) respectively.

![Graph](image)

Figure 2.10: Front surface distortion predicted by the simulations (solid lines), the model 2.31-2.34 (dotted lines) and the equation 2.36 (dashed lines) for two rear surface perturbations with \( kd_{ps} = 0.1 \) and \( kd_{ps} = 0.5 \).

Figure 2.10 shows the time evolution of the front surface distortion \( \Delta_{fs} \) obtained from the model 2.31-2.34, the approximate formula 2.36 and the numerical simulations for wavelengths \( \lambda = 60\mu m \) (\( kd_{ps} = 1/2 \)) and \( \lambda = 300\mu m \) (\( kd_{ps} = 1/10 \)). The functions \( g_{fs}(t) \) and \( \delta g_{fs}(t) \) are obtained from 2.21,2.26 as long as the flow is isentropic (or almost isentropic) until \( \sim 1.2\eta s \), and from the steady state values after that. The good agreement between the analytic theory and simulations for \( kd_{ps} = 1/2 \) shows that, in spite of the \( kd_{ps} < 1 \) expansion, the analytic theory is also valid for finite \( kd_{ps} \). In order to further test the analytic mode, we have carried out a detailed comparison of the solution of 2.28 and the numerical simulation.
during the initial phase of the RT growth when the perturbation does not grow exponentially. A long wavelength mode with $\lambda = 300 \mu m$ begins to grow exponentially after a time $t > t_{rb} + 1/\sqrt{kg} \approx 1.3 \text{ns}$. Before then, we expect the distortion to be driven first by the imprinted velocity perturbations and grow linearly in time. Furthermore since $\delta g$ is negative right after the rarefaction wave breaks out (see figure 2.9), we expect the linear growth to slow down immediately. Later in time, $\delta g$ becomes positive and a quadratic growth is expected.

![Ablation-front-surface distortion versus time](image)

Figure 2.11: Time evolution of the front surface distortion induced by a rear surface perturbation with wavelength $\lambda = 300 \mu m$. The solid lines represent analytical and numerical results. The dashed line shows the initial derivative of the perturbation.

Figure 2.11 compares the front surface distortion induced by a $300 \mu m$ wavelength perturbation calculated from 2.28 and the results of the numerical simulation. The dashed line represents a distortion developing linearly in time with the imprinted velocity perturbation. As predicted, the distortion level decays below
the linear growth immediately after the rarefaction wave breaks out and later turns into the quadratic growth. Also observe that the analytic model and the numerical simulations are in excellent agreement throughout this initial stage confirming the validity of the analytic model.

Finally, we have carried out several simulations for different mode wavelengths and numerically determined the transfer function $F$. The target is the same planar foil as described above and the perturbation wavenumber ranges from $kd_{ps} \simeq 0.01$ up to $kd_{ps} \simeq 10$. Figure 2.12 shows the comparison between the numerical simulations and the analytic formula 2.37. The agreement between theory and simulations persists up to $kd_{ps} \simeq 1$. At short wavelengths ($kd_{ps} > 1$), the theory breaks down and the numerical results develop a complicated behavior. It is important to remember that the code used in such simulations does not include heat transfer or mass ablation. As shown in the next section, both processes reduce the growth of short wavelength modes and significantly affect the transfer function.

2.6 Short Wavelength Feedout.

It is well known that the time evolution of short wavelength modes is significantly affected by thermal conductivity, finite density gradient scale length and mass ablation. As discussed in Chapter 1, the RT growth rate itself is reduced when $kL_m \sim 1$ and/or $kV_a \sim \sqrt{kg}$ and the short wavelengths instability is suppressed. It is still meaningful to determine the transfer function $F$ for short wavelengths based on the definition adopted for long wavelength modes

$$\Delta_{fs} \simeq \Delta_0 F e^{\int_{t_b}^{t} \gamma dt'}$$  \hspace{1cm} (2.38)
Figure 2.12: The transfer function $F$ versus the dimensionless wavenumber $kd_{ps}$. The solid line is the analytic formula 2.37, and the dots are the results of the numerical simulations performed without heat transfer. The agreement is good as long as $kd_{ps} < 1$. Both the curve and the dots corresponding to $kd_{ps} > 1$ do not represent the transfer function $F$ properly because mass ablation is not included in both theory and simulations.
where $\Delta f_s$ is the front surface distortion and $\Delta_0$ is the initial rear surface perturbation. However, it is important to emphasize that the growth rate $\gamma$ must include the effects of ablation, thermal conduction smoothing and finite density gradient scale length. For such a reason, we use the equation 2.38 with the growth rate given in [10] which includes all these stabilizing effects. Because of the many physical mechanisms affecting the short wavelength feedout, the mathematical complexity of an analytic solution becomes overwhelming. A numerical solution of the problem seems to be the best option to explore the short wavelength regime.

The two dimensional Lagrangian code is described in section 4.1 and makes use of explicit differencing of the hydrodynamic equations and an implicit thermal transport solver.

Several numerical simulations of the short wavelength feedout in plastic foils have been performed. Plastic is the standard material used in the hydrodynamic stability experiments at LLNL and LLE. We have considered foils with a $\sim 20\mu m$ thickness driven by a constant laser intensity. The laser intensity is sufficiently large to drive a strong uniform shock through the target so that the post shock thickness is about $5\mu m$. Initially, the front surface is perfectly uniform while the rear surface is distorted by a single mode perturbation in the linear regime (see figure 2.13). Mode wavelengths ranging from 4 to $20\mu m$ have been considered.

We first compare the behavior of short wavelength modes with the one of long wavelengths. Figure 2.14 compares the evolution of a $20\mu m$ and a $4\mu m$ wavelength mode. Observe that the $4\mu m$ wavelength mode features an oscillatory behavior before the exponential growth. This is not surprising because the perturbation induced on the isobars within the rarefaction wave oscillates in space. This can also be deduced from the analytic solution by A.Velikovich et al. describing a perturbed rarefaction wave originating from a rippled surface. It is shown in [40] that
Figure 2.13: The initial shape of the simulated planar targets. The back surface is rippled, while the front surface is flat. Laser light illuminates the front surface launching the shock. The laser intensity profile is a step function.

The perturbed pressure inside the rarefaction fan oscillates like a Bessel function along the direction of propagation. Thus, we conjecture that the ablation front oscillations seen in figure 2.14 are caused by the different isobars of the rarefaction fan reaching the front surface and imprinting the velocity perturbations with opposite phases.

The next step consists in carrying out the simulations for different rear surface perturbation wavelengths and focusing on their exponential growth during the acceleration phase. Such results together with the analytic growth rate formula of [10] are then used to determine the transfer function $F$ in the equation 2.38. The actual form of the transfer function is obtained by fitting the numerical results using a simple algebraic formula dependent on the relevant dimensionless parameters. The numerical results indicate that deviation of the transfer function from its classical form 2.37 in logarithmic scale is quadratic in the dimensionless parameter.
Figure 2.14: Evolution of the front surface perturbations obtained in the numerical simulation of the feedout. The short-wavelength mode oscillates, while the long-wavelength mode does not. An exponential growth is observed in both cases.

\[ \frac{\Delta s(t)}{\Delta 0} \]

\[ \text{Time, ns} \]

\[ \text{Long wavelength mode (20\,\mu m)} \]

\[ \text{Short wavelength mode (4\,\mu m)} \]

\[ kd \] (see figure 2.15); i.e. the transfer function can be written in the following form:

\[ \log (F) = \log (F_c) \left\{ 1 + \alpha (kd)^2 \right\} \]

(2.39)

where \( F_c \) is the classical transfer function 2.37 and \( \alpha \) is a parameter which depends on the remaining dimensionless parameters.

The other dimensionless parameters can be derived by inspection of the conservation equations 2.2-2.6. We assume here that the only parameters characterizing the process are the wavenumber \( k \), acceleration \( g \), post-shock sound speed \( a_{ps} \), density gradient scale length \( L_m \), ablation velocity \( V_a \), and the shell thickness \( d \) (\( d \) stands for compressed shell thickness in this section). According to the Pi theorem of dimensional analysis (see, for example, [41]), 4 independent dimensionless parameters can be constructed. Besides the Mach number which is typically small
Figure 2.15: The dots represent the transfer function $F$ obtained in numerical simulations of short-wavelength feedout. The deviation of the numerical results from the long-wavelength theory (solid line) can be described by the fitting formula 2.39 (dashed line). The best fit is obtained using the least square method.

(and therefore negligible) and the dimensionless wavenumber $kd$ (already in use), the other parameters are the Froude number

$$Fr = \frac{Va^2}{gL_0} \quad (2.40)$$

and the dimensionless density gradient scale length

$$L_m/d \quad (2.41)$$

where $L_m$ is the minimum density gradient scale length:

$$L_m = \rho \left( \frac{\partial \rho}{\partial y} \right)^{-1}, \quad L_0 = L_m \frac{\nu^\nu}{(\nu + 1)^{\nu + 1}}, \quad (2.42)$$

$\nu$ is the power index for thermal conduction, $Va$ is the ablation velocity and $g$ is the acceleration.
We have investigated the dependence of the parameter $\alpha$ on the dimensionless parameters within the range $0.02 < L_m/d < 0.52$, $0.01 < Fr < 0.17$. Such parameter space was accessed by varying the laser intensity and thermal conductivity. The two parameters lie in a narrow stripe in the $(L_m/d, Fr)$ plane (figure 2.16), thus suggesting that they are not independent.

![Figure 2.16: Variation of laser intensity and thermal conductivity coefficient gives the set of points lying within a narrow strip in the $(L_m/d, Fr)$ plane.]

Since the dependence $\alpha(Fr)$ is ambiguous for $Fr < 0.06$, the parameter $L_m/d$ was chosen as the independent parameter. The set of points $\alpha(L_m/d)$ and the best linear fit to the data are represented in figure 2.17.

Substituting the best fit coefficients into 2.39, one finds the following fitting formula for the transfer function $F$:

$$\log (F) = \log (F_c) \left\{ 1 + 0.005 \left( kd \right)^2 + 0.038 \left( kL_m \right) \left( kd \right) \right\}, \quad (2.43)$$

where $F_c$ is the long wavelength form of the transfer function given in 2.37:

$$F = \frac{1}{16} \left( \frac{1}{kd_{ps}} + \frac{2.5}{\sqrt{kd_{ps}}} \right), \quad (2.44)$$
Figure 2.17: The solid line is the best linear least square fit to the data $\alpha (L_m/d_{ps})$. Each point represents the coefficient $\alpha$ (see 2.39) evaluated for a given laser intensity and heat conductivity.
Equation 2.43 includes the finite $kd$ corrections to 2.37, and it can be used to determine the RT seeding by the feedout of short as well as long wavelength perturbations. The dashed line in figure 2.15 represents the result 2.43 and reproduces the data obtained numerically (dots).
Chapter 3

Deceleration Phase Instability.

3.1 Introduction.

The deceleration phase RT instability is the instability of the inner surface of the shell occurring when the shell is decelerated by the large hot-spot pressure. Driven by the instability, fingers of cold shell materials penetrate into the hot-spot leading to significant cooling and quenching the ignition process. Furthermore, if the inner surface perturbation becomes nonlinear, a fraction of the shell kinetic energy is used to feed the lateral shell motion induced by the instability, reducing the compression work done by the shell on the hot-spot. Typical seeds for the deceleration phase instability are the outer surface nonuniformities feeding through the shell during the acceleration phase.

It is common wisdom (see, for example, [2]) that the deceleration phase RT instability is classical and all modes are unstable with growth rates

$$\gamma(k) \approx \sqrt{\frac{kg}{1 + kL_m}}$$

(3.1)

where $g$ is the shell deceleration, $L_m$ is the density gradient scale length and $k$
Figure 3.1: Temperature and density profiles at stagnation. Ablation from the inner surface takes place due to the thermal expansion of the material heated up by the thermal flux from the hot-spot.

is the perturbation wave number approximately equal to $l/R$ with $R$ being the hot-spot radius and $l$ the Legendre mode number. Observe that according to the equation 3.1 all modes are unstable, and the growth rate approaches an asymptotic value $\gamma(kL_m \gg 1) \approx \sqrt{g/L_m}$ as $k \to \infty$.

Section 3.2 of this chapter is devoted to the development of a one-dimensional model describing the shell deceleration and hot-spot ignition. The model is valid from the beginning of the continuous deceleration phase (after the shock reflection off the shell) until the burn wave propagation in the shell which is not included in the calculation. The evolution of the hot-spot radius, mass, temperature, density and pressure are calculated in terms of the initial conditions at the beginning of the continuous deceleration phase. Furthermore, a two dimensional diagram is derived.
showing how the hot-spot ignition depends only on two dimensionless parameters related to the initial conditions. The section 3.3 deals with the hydrodynamic stability analysis of the shell during the continuous deceleration phase. It is shown that mass ablation from the shell inner surface significantly reduces the deceleration RT growth rates, leading to much lower growth rates than predicted by equation 3.1 and to a cutoff in the unstable spectrum. Mass ablation is caused by the heat flux leaving the hot-spot and depositing on the shell inner surface. We have calculated the ablation velocity and the shell density-gradient scale length during the deceleration phase. Then, using the RT theory developed in [10, 12], we have calculated the growth rates and compared them with the results of numerical simulations. For direct-drive NIF-like capsules, the cutoff mode number for the DPRT is approximately $l_{cutoff} \approx 90$.

3.2 Hot-Spot Dynamics.

The hot-spot is a low density plasma heated by the shock and by the compression work of the cold dense surrounding shell. Its dynamics is governed by the mass, momentum, and energy conservation equations. The energy equation must include the electronic thermal conduction and alpha particle energy deposition. Bremsstrahlung radiation energy losses are neglected in this model because they add great complexity to the mathematical solution and their contribution is typically smaller than the mechanical work and/or the fusion power. The magnitude of the radiation losses is larger than the fusion power for temperatures below 4.4keV when the $PdV$ work rate is typically greater than both radiation and fusion power. Thus, at such low temperatures, both radiation losses and fusion power are negligible with respect to the compression work rate. The $PdV$ work rate decreases
near the shell stagnation time where higher temperatures are reached within the hot-spot. If such temperatures are well above 4.4keV then the $\alpha$-particle power is greater than the radiation losses and the bremsstrahlung term can again be neglected in the energy equation. The model described in this section may not apply to capsules that do not ignite or with small ignition margins (small mechanical work rate and low final temperatures) as their evolution can be significantly affected by radiation losses.

This model is expected to predict the main characteristics of the deceleration phase up to the onset of ignition. The actual ignition process in the hot-spot is not accurately modeled because the $\alpha$-particle energy is assumed to be locally deposited within the hot spot. Instead, the $\alpha$-particle mean free path is typically of the same order of magnitude as the hot-spot radius and a fraction of them leaves the hot-spot and is deposited on the shell inner surface. Those particles trigger the propagation of a burn wave in the cold dense shell which burns until it disassembles. The thermonuclear burn wave propagation is not described by our model and requires a diffusion or kinetic treatment of the alpha particle population.

Even though the local deposition approximation used here is strictly valid only when all the $\alpha$-particles are absorbed within the hot-spot, we artificially include the effect of alpha particle diffusion by adding a multiplicative factor $\theta \leq 1$ to the alpha heating term. When $\theta < 1$, one should also include the $\alpha$-particle power deposited at the inner shell surface due to the $1 - \theta$ fraction of leaking particles. Such a contribution is not included in the derivation of the hot-spot profiles, ablation velocity and density gradient scale length because an analytic solution of the conservation equations could not be found. Thus, an additional limitation of this model is that the fraction of alpha particles leaving the hot-spot is small compared to the fraction absorbed.
In conclusion, the model given below is suitable to describe the deceleration phase up to the onset of ignition but it does not include the relevant physics pertaining to the ignition process or the burn wave propagation. We speculate that most of the RT instability growth occurs before ignition takes place when our model captures the essential physics of the hot-spot dynamics.

We consider the 1D compressible fluid equations in spherical geometry and define the Lagrangian coordinate system \((\bar{r}, t')\), related to the Eulerian system \((r, t)\) in the following way:

\[
\begin{align*}
  r &= \bar{r} + \int_0^t u(\bar{r}, t') \, dt' \\
  t &= t'
\end{align*}
\]  

(3.2)  

(3.3)

The Lagrangian coordinate \(\bar{r}\) thus corresponds to the initial position of a fluid element. The equations 2.2-2.5 then become:

\[
\begin{align*}
  \rho r^2 \, dr &= \rho_0 \bar{r}^2 \, d\bar{r} \\
  \frac{\partial}{\partial t} u &= -\frac{1}{\rho_0} \frac{r^2}{\bar{r}} \frac{\partial P}{\partial \bar{r}} \\
  c_v \rho^{\Gamma-1} \frac{\partial}{\partial t} \frac{T}{\rho_0 r^2 \bar{r}^{\Gamma-1}} &= \frac{1}{\rho_0} \frac{\partial}{\partial \bar{r}} \frac{k}{\rho_0 \bar{r}^2} \frac{\partial T}{\partial \bar{r}} + \frac{\rho}{4m_i^2} \theta E_\alpha \langle \sigma v \rangle \\
  P &= \frac{\rho T}{A}
\end{align*}
\]  

(3.4)  

(3.5)  

(3.6)  

(3.7)

where \(\rho_0\) is the initial density of the fluid, \(\Gamma\) is the ratio of specific heats \((\Gamma = 5/3\) for monoatomic gases), \(\kappa = k(T)\) is the thermal conductivity, \(T\) is the temperature, \(c_v = 1/(A(\Gamma - 1))\), \(c_v\) is the specific heat under constant volume, \(A = m_i/(1 + Z)\), \(m_i\) and \(Z\) are the ion mass and atomic number, respectively \((Z = 1\) for DT), \(\langle \sigma v \rangle\) is the fusion reaction rate, \(E_\alpha = 3.5\text{MeV}\).
Introducing the independent variable $m$, proportional to the mass within the radius $r$

$$m = \int_0^r \rho(r')r'^2 \, dr' = \int_0^r \rho_0(r')r'^2 \, dr'$$  \hspace{1cm} (3.8)

(note that $dm = \rho r'^2 \, dr = \rho_0 r'^2 \, dr$), one can rewrite the set of fluid equations in terms of the variables $(m, t)$:

$$\frac{\partial}{\partial t} u = -r^2 \frac{\partial P}{\partial m}$$  \hspace{1cm} (3.9)

$$c_s^2 \rho^{\Gamma - 1} \frac{\partial}{\partial t} \left( \frac{T}{\rho^{\Gamma - 1}} \right) = \frac{\partial}{\partial m} \kappa(T) \rho r^4 \frac{\partial T}{\partial m} + \frac{\rho}{4m_i^2} \theta E_{\alpha}(\sigma\nu)$$  \hspace{1cm} (3.10)

We assume that the hydro flow is subsonic flow and order $\epsilon \sim u/c_s \ll 1$. The typical distance in the problem is $r \sim R_{\text{hot-spot}}$, the typical time scale is $t \geq R_{\text{hot-spot}}/c_s$. So the order of terms in the momentum equation 3.5 can be roughly estimated as

$$\frac{\partial u}{\partial t} \sim \frac{\epsilon c_s}{R_{\text{hot-spot}}/c_s} \sim \frac{\epsilon c_s^2}{R_{\text{hot-spot}}}$$  \hspace{1cm} (3.11)

$$r^2 \frac{\partial P}{\partial m} \sim \frac{R_{\text{hot-spot}} P}{\rho R_{\text{hot-spot}}^2 R_{\text{hot-spot}}} \sim \frac{c_s^2}{R_{\text{hot-spot}}}$$  \hspace{1cm} (3.12)

Therefore the pressure $P = P_0 + P_1 + \ldots$, $P_1 \sim \epsilon$,

$$\frac{\partial P_0}{\partial m} = 0, \quad r^2 \frac{\partial P_1}{\partial m} = \frac{\partial u}{\partial t}, \ldots$$  \hspace{1cm} (3.13)

and to the lowest order of $\epsilon$

$$P = P_0 = P_0(t)$$  \hspace{1cm} (3.14)

is the function of time only, reproducing the so called flat pressure approximation.

The fusion rate can be approximated with a quadratic term $\langle \sigma v \rangle = \Sigma_\alpha T^2$, $\Sigma_\alpha \approx 1.05 \times 10^{-24} m^3 s/keV^2$. As shown in figure 3.2, the quadratic approximation is valid as long as $4 < T < 20 keV$, which is the temperature range relevant to the ICF ignition experiments.
Substituting the ideal gas equation of state 3.7 into the energy equation 3.6 and assuming power dependence of the heat conductivity on the temperature \( \kappa(T) = \kappa T^\nu \), one obtains the following equation:

\[
c_v \left( \frac{P}{T} \right)^{\Gamma - 1} \frac{\partial}{\partial t} \left( \frac{T^\Gamma}{P^{\Gamma - 1}} \right) = A\kappa P \frac{\partial}{\partial m} T^{\nu - 1} r^4 \frac{\partial T}{\partial m} + \frac{\theta E_{\alpha \Sigma} A}{m_i^2} PT
\]

(3.15)

We define the new function \( \Pi(m, t) \):

\[
\Pi(m, t) = \frac{T(m, t)}{P(t)^{\Gamma - 1}}, \quad \text{or} \quad T(m, t) = \Pi(m, t) P(t)^{\Gamma - 1}
\]

(3.16)

and the new independent variable

\[
\tau \equiv \tau_0 + \frac{3^{4/3} \kappa}{A^{1/3} c_v} \int_0^t [P(t')]^\beta dt', \quad \beta = \frac{3\nu(\Gamma - 1) - 1}{3\Gamma}
\]

(3.17)

where \( \tau_0 \) is an arbitrary constant to be determined by the initial conditions. Given \( \nu = 5/2 \) (Spitzer conductivity) and \( \Gamma = 5/3 \) (monatomic gas), one obtains \( \beta = 4/5 \).

We use the definition 3.8 and the equation of state 3.7 to determine the relation between the volume within the radius \( r \) and the function \( \Pi \):

\[
dm = \rho r^2 dr = \frac{\rho}{3} d \left( r^3 \right), \quad d \left( r^3 \right) = \frac{3 dm}{\rho} = \frac{3T dm}{AP}
\]

(3.18)
\[ r^3 = \frac{3}{AP(t)} \int_0^m T(m')dm' = \frac{3}{AP} \int_0^m \Pi P^{\frac{r-1}{r}} dm' \]
\[ = \frac{3}{AP} \int_0^m \Pi dm' = \frac{3}{AP} \Phi \]  
(3.19)

where the function \( \Phi(m, t) \) was introduced as

\[ \Phi(m, t) \equiv \int_0^m \Pi(m', t)dm' \]  
(3.20)

The energy equation 3.15 in the new variables can be written as

\[ \frac{\partial}{\partial r} \Pi = \frac{\partial}{\partial m} \Pi^\nu \Phi^{4/3} \frac{\partial}{\partial m} \Pi + \Delta_{\alpha} \Pi P(t)^\delta \]  
(3.21)

or, after the substitution \( \Pi = \partial \Phi / \partial m \) and integrating between 0 and \( m \):

\[ \frac{\partial}{\partial \tau} \Phi = \Phi^{4/3} \left( \frac{\partial \Phi}{\partial m} \right)^\nu \frac{\partial^2 \Phi}{\partial m^2} + \Delta_{\alpha} \Phi P(t)^\delta \]  
(3.22)

where

\[ \delta = \frac{3\nu + 1 + 3\Gamma(1 - \nu)}{3\Gamma}, \quad \Delta_{\alpha} = \left( \frac{A}{3} \right)^{\frac{3}{2}} \frac{\theta E_{\alpha} \Sigma_{\alpha}}{4\kappa m_i^2} \]  
(3.23)

Observe that the \( \alpha \)-particle term on the right hand side of 3.22 can be combined with the left hand side by defining the new dependent (\( \Psi \)) and independent (\( \eta \)) variables

\[ \Psi = \Phi \exp \left[ -D_{\alpha} \int_0^t P(t')dt' \right], \]  
(3.24)

\[ \eta = \eta_0 + \frac{3^{4/3}\kappa}{A^{1/3}\Gamma c_0} \int_0^t dt' P(t') \exp \left[ (\nu + \frac{1}{3})D_{\alpha} \int_0^{t'} P(t'')dt'' \right], \]  
(3.25)

where

\[ D_{\alpha} = \frac{\Gamma - 1}{4\Gamma} \frac{\theta E_{\alpha} \Sigma_{\alpha}}{(1 + Z)^2} \]  
(3.26)

and \( \eta_0 \) is a new constant. After a short calculation, 3.22 can be rewritten in the following simple form:

\[ \frac{\partial \Psi}{\partial \eta} = \Psi^{4/3} \left( \frac{\partial \Psi}{\partial m} \right)^\nu \frac{\partial^2 \Psi}{\partial m^2} \]  
(3.27)
The solution of the equation 3.27 can be sought as a function of a self-similar dimensionless variable $\xi$:

$$\xi = \frac{am}{\eta^\alpha}$$  \hspace{1cm} (3.28)

where $\Omega$ is the dimensionless parameter to be determined, and $\alpha$ is the parameter introduced to keep the variable $\xi$ dimensionless.

Look for a self-similar solution of the equation 3.27:

$$\Psi(\eta, m) = \frac{1}{\alpha^\frac{3(\nu+1)}{3\nu+1} \eta^\frac{3(1-\Omega(\nu+1))}{3\nu+1}} F(\xi)$$  \hspace{1cm} (3.29)

After the substitution of 3.29 into 3.27 one obtains:

$$F^{4/3} \left( \frac{dF}{d\xi} \right)^{\nu-1} \frac{d^2 F}{d\xi^2} + \Omega \xi \frac{dF}{d\xi} + \frac{3[1 - \Omega(\nu + 1)]}{3\nu + 1} F = 0$$  \hspace{1cm} (3.30)

Since the hot-spot temperature is much larger then the cold shell temperature, we apply the zero temperature boundary condition at $r = R_{\text{hotspot}}$:

$$T(R_{\text{hotspot}} = 0$$  \hspace{1cm} (3.31)

$$\Pi(R_{\text{hotspot}} = 0$$  \hspace{1cm} (3.32)

$$\frac{\partial \Phi}{\partial m} \bigg|_{r=R_{\text{hotspot}}} = 0$$  \hspace{1cm} (3.33)

$$\frac{dF}{d\xi} \bigg|_{r=R_{\text{hotspot}}} = 0$$  \hspace{1cm} (3.34)

Thus, at $r = R_{\text{hotspot}},$

$$\frac{3[1 - \Omega(\nu + 1)]}{3\nu + 1} F(\xi_0) = 0$$  \hspace{1cm} (3.35)
where \( \xi_0 \) corresponds to the hot-spot radius, \( F(\xi_0) \neq 0 \) for any non-trivial solution, and so
\[
\Omega = \frac{1}{\nu + 1}
\]  
(3.36)

and equation 3.30 becomes
\[
\frac{1}{\nu + 1} \xi + F^{4/3} \left( \frac{dF}{d\xi} \right)^{\nu - 2} \frac{d^2 F}{d\xi^2} = 0
\]  
(3.37)

The equation 3.37 can be numerically solved, given the set of initial conditions. Since \( F(\xi) \) is proportional to the energy associated with the mass \( m \), it follows that
\[
F(0) = 0
\]  
(3.38)

Observe that the self-similar variable \( \xi \) depends the arbitrary constant \( a \) and the derivative \( dF/d\xi \) depends on \( a \). Thus, any choice of the derivative would only affect the constant \( a \), while all physical quantities are independent on \( F'(0) \), so that the second boundary condition can be chosen as:
\[
\left. \frac{dF}{d\xi} \right|_{\xi=0} = 1
\]  
(3.39)

The numerical solution (see figure 3.3) found for \( \nu = 5/2 \), gives \( dF/d\xi = 0 \) at \( \xi = \xi_0 = 1.23 \) and \( F(\xi_0) = 0.7 \).

Since the temperature (proportional to \( F(\xi) \)) vanishes at \( \xi_0 \), we use \( \xi = \xi_C \) as the outer boundary of the hot spot and set
\[
m_{hs} = \frac{\xi_0 \eta^{\frac{\nu+1}{\nu}}}{a},
\]  
(3.40)

where \( m_{hs} \) is the value of coordinate \( m \) corresponding to the hot-spot radius.

The hot-spot mass is:
\[
M_{hs} = \int_0^{R_{hs}} \rho 4\pi r^2 \, dr = 4\pi \int_0^{m_{hs}} \, dm = 4\pi m_{hs}
\]  
(3.41)
Figure 3.3: Plot of functions $F(\xi)$ and $F'(\xi)$ obtained from numerical solution of the equation 3.37.

\[ M_{hs} = 4\pi \xi_0 \frac{\eta_0^{1+1}}{a} \] (3.42)

The constants $a$ and $\eta_0$ can be determined from the initial conditions applied to equations 3.42, 3.29, and 3.19 leading to

\[ a = \left[ \frac{3F(\xi_0)}{AR_{hs}(0)^3 P(0)^{1/\Gamma}} \right]^{\frac{\nu+1}{3(\nu+1)}}; \quad \eta_0 = \left[ \frac{aM_{hs}(0)}{4\pi \xi_0} \right]^{\nu+1} \] (3.43)

where $R(0)$, $P(0)$ and $M_{hs}(0)$ are the initial hot-spot radius, pressure and mass respectively. A short calculation using 3.19, 3.24, 3.29, 3.42 and the equation of state 3.7 yields the relevant hot-spot parameters (mass, areal density $\rho R$, density and temperature) as functions of the hot-spot pressure and radius:

\[ M_{hs}(t) = \left\{ M_{hs}(0)^{\nu+1} + \chi_0 R A^{\nu+1} \int_0^t P(t')^{\beta} \left[ R_{hs}(t')^3 P(t')^{\frac{1}{3}} \right]^{\nu+\frac{1}{3}} dt' \right\}^{\frac{1}{\nu+1}} \] (3.44)

\[ \rho R \equiv \int_0^{R_{hs}} \rho dr = \mu_0 \frac{M_{hs}(t)}{4\pi R_{hs(t)}^2}, \] (3.45)

\[ \rho_{hs}(m, t) = \frac{3M_{hs}(t)}{4\pi R_{hs(t)}^3} \frac{F(\xi_0)}{\xi_0 F'(\xi)}, \quad T_{hs}(m, t) = \frac{AP(t)}{\rho_{hs}(m, t)}, \] (3.46)

where

\[ \chi_0 \equiv \frac{\Gamma - 1}{\Gamma} \frac{1}{3^{1-\nu}} \left( 4\pi \xi_0 \right)^{\nu+1} \left( F(\xi_0)^{\nu+\frac{1}{3}} \right), \quad \mu_0 = \frac{F(\xi_0)^{2/3}}{\xi_0} \int_0^{\xi_0} \frac{d\xi}{F(\xi)^{2/3}}. \] (3.47)
Observe that the hot-spot mass increases with time at the rate which depends on the thermal conductivity coefficient. The mass increase is due to the ablation of the shell material into the hot-spot. The hot-spot radius and pressure are related through 3.19 which can be rewritten by using the initial conditions leading to

\[
\frac{R_{hs}(t)^3}{R(0)^3} = \left[ \frac{P(0)}{P(t)} \right]^{1/T} \exp \left[ D_\alpha \int_0^t P(t) \right]
\]  

(3.48)

It is important to notice that the equation 3.48 is a straightforward consequence of the hot-spot global energy conservation equation. This can be shown by writing the total (internal+kinetic) energy equation in the conservative form

\[
\frac{\partial}{\partial t} \left( \frac{P}{\Gamma - 1} + \rho \frac{U^2}{2} \right) + \nabla \cdot \left[ \bar{u} \left( \frac{\Gamma P}{\Gamma - 1} + \rho \frac{U^2}{2} \right) \right] = \nabla \cdot \kappa(T) \nabla T - \frac{\rho}{4m_i^2} \theta E_\alpha \langle \sigma \nu \rangle
\]  

(3.49)

Using the subsonic flow assumption \( U \ll c_s \), we neglect the kinetic energy with respect to the internal energy. Then, after approximating the fusion cross section with the quadratic form \( \langle \sigma \nu \rangle \simeq \Sigma_\alpha T^2 \), equation 3.49 is integrated over the hot-spot volume enclosed by the inner shell surface. At the surface, the shell material is cold and the thermal conduction can be neglected (\( \kappa \approx 0 \)). A straightforward calculation leads to the following form of the energy equation:

\[
\frac{d}{dt}(P_{hs} R_{hs}^3) + 3P_{hs}^2 R_{hs} \left[ \Gamma U(R_{hs}, t) - \frac{dR_{hs}}{dt} \right] = \Gamma D_\alpha P_{hs}^2 R_{hs}^3
\]  

(3.50)

where \( U(R_{hs}, t) \) is the flow velocity at the shell inner surface. The flow velocity is the combination of the inner surface motion and the ablative flow,

\[
U(R_{hs}, t) = \dot{R}_{hs} - V_a
\]  

(3.51)

where \( V_a \) is the ablation velocity and \( \dot{R}_{hs} \) is the velocity of the inner surface. Since \( V_a \ll \dot{R}_{hs} \), the ablation velocity can be neglected and 3.50 yields the exact solution 3.48. Thus, 3.48 and 3.50 are the equivalent forms of the energy equation.
Notice that the heat conduction losses do not enter into the global energy balance of the hot-spot. This is because the heat flux leaving the hot-spot is deposited onto the inner shell surface. A fraction of this energy is transformed into internal energy of the shell material ablating into the hot-spot. The remaining fraction produces the $PdV$ work done by the ablated plasma entering the hot-spot against the hot-spot pressure. In other words, the energy leaving the hot-spot in the form of heat conduction losses goes back into the hot spot in the form of internal energy and compression work of the ablated plasma. Therefore, conduction losses are not real energy losses and do not affect the global energy balance of the hot-spot as shown by the equation 3.50. It is important to emphasize that the hot spot energy is proportional to its pressure. The conduction losses do affect the hot-spot temperature but not its pressure. This conclusion implies that greater heat conduction losses would lower the temperature and raise the density (through larger ablation at the shell inner surface) leaving the pressure ($P \sim \rho T$) unaltered.

Equation 3.48 relates hot-spot radius and pressure. All the hydrodynamic quantities shown in 3.44-3.47 can therefore be expressed as functions of the pressure only. It follows that a fully self-consistent implosion model requires additional equations relating $P(t)$ to the shell properties. A shell model used to determined $P(t)$ is described in Section 3.4.

### 3.3 Ablation Velocity and Density Gradient Scale Length.

An important result of the analysis carried out in the previous section concerns the hot-spot mass. Equation 3.44 shows that the hot-spot mass increases with time. Its
rate of increase depends on the magnitude of the heat conduction coefficient $\kappa$ and the hot-spot pressure. The ablation velocity at the inner shell surface follows by noticing that the mass ablation rate off the shell $\dot{M}_a$ must equal the rate of change of the hot-spot mass $\dot{M}_{hs}$. Given the hot-spot radius $R_{hs}$ and the shell peak density $\rho_{shell}$, the ablation rate is $\dot{M}_a = 4\pi R_{hs}^2 \rho_{shell} V_a$, where $V_a$ is the ablation velocity. Thus setting $\dot{M}_a = \dot{M}_{hs}$ yields the ablation velocity

$$V_a = \frac{\dot{M}_{hs}}{4\pi R_{hs}^2 \rho_{shell}}, \quad (3.52)$$

where $\dot{M}_{hs}$ can be determined from 3.44. Then, using the $m$-derivative of $\Phi$ to relate $T$ and $\eta$:

$$T = P \frac{\partial \Phi}{\partial m} = P \frac{r_{cv}^1}{\Gamma} a^{-\frac{3(\nu+1)}{\nu+1}} \frac{df}{d\xi} \frac{a}{\eta^{\nu+1}} \quad (3.53)$$

$$T_0 = T(m = 0, t) = P \frac{r_{cv}^1}{\Gamma} a^{-\frac{3(\nu+1)}{\nu+1}} \eta^{-\nu+1} F'(0) \exp \left[ D_\alpha \int_0^t P(t') dt' \right] \quad (3.54)$$

$$\eta^{-\nu+1} = \frac{P \frac{r_{cv}^1}{\Gamma}}{T_0} F'(0) a^{-\frac{3(\nu+1)}{\nu+1}} \exp \left[ D_\alpha \int_0^t P(t') dt' \right], \quad (3.55)$$

one can obtain the ablation velocity in terms of standard hot-spot and shell parameters

$$V_a = \frac{3(\Gamma - 1)}{(\nu + 1)\Gamma} \frac{\xi_0}{F'(0)^{\nu} F(\xi_0)^{1/3}} \frac{A\kappa T_{hs}(0, t)^{\nu}}{\rho_{shell}(t) R_{hs}(t)}, \quad (3.56)$$

where both the central hot-spot temperature and radius depend only on the hot-spot pressure. From the numerical solution of 3.37 with $F'(0) = 1$ and $\nu = 5/2$, $\xi_0 = 1.23$, $F(\xi_0) = 0.7$. The ablation velocity can then be calculated using standard ICF units leading to the following simple form

$$V_a(\mu m/ns) = 6 \cdot 10^3 \frac{(T_{hs}^{\text{kev}})^{5/2}}{R_{hs}^{\text{km}} \rho_{shell}^{\text{g/cm}^3} \Lambda_{hs}}, \quad (3.57)$$

where $R_{hs}$, $T_{hs}$, $\Lambda_{hs}$ are the hot-spot radius, central temperature, and Coulomb logarithm, and $\rho_{shell}$ is the shell peak density.
In addition to the ablative stabilization, the RT growth rates are reduced by the well known finite density-gradient effects. Since the ablative flow at the inner shell surface is subsonic, the minimum density gradient scale length can be calculated using the well known isobaric model (see [15, 12]) characterized by the following approximate form of the energy equation

$$\nabla \cdot \left[ \bar{\tau} \frac{\Gamma P}{\Gamma - 1} - \kappa \nabla T \right] = 0$$ \hspace{1cm} (3.58)

Integrating 3.58 and using the continuity of the mass flow ($\rho U = \text{constant}$) leads to the following ordinary differential equation for the density profile near the inner shell surface

$$\frac{1}{\bar{\rho}} \frac{\partial \bar{\rho}}{\partial r} = \bar{\rho}^\nu \left(1 - \bar{\rho}\right) \frac{1}{L_0}$$ \hspace{1cm} (3.59)

where $\bar{\rho} = \rho / \rho_{\text{shell}}$ is the density normalized to the peak density in the shell ($\rho_{\text{shell}}$) and

$$L_0 = \frac{\Gamma - 1}{\Gamma} \frac{A_\kappa(T_{\text{shell}})}{\rho_{\text{shell}} V_a}$$ \hspace{1cm} (3.60)

Here $T_{\text{shell}} = AP(t)/\rho_{sh}$ represents the temperature calculated at the peak of the density. The minimum value of the density gradient scale length $L_m$ can be determined by setting to zero the radial derivative of 3.59. A straightforward manipulation yields

$$L_m = L_0 \frac{(\nu + 1)\nu^{\nu+1}}{\nu^\nu}$$ \hspace{1cm} (3.61)

Using $\nu = 5/2$ and substituting 3.57 and 3.60 into 3.61 leads to the following simple expression of the density gradient scale length

$$L_m = 6.8R_{hs} [AP(t)/\rho_{\text{shell}} T_{hs}(0, t)]^{5/2}$$ \hspace{1cm} (3.62)

where $P(t)$ is the hot-spot pressure.

This conclude the hot-spot analysis. All the relevant hot-spot parameters have been calculated as functions of the hot-spot pressure. The next step is to determine
the evolution of the hot-spot pressure by coupling the hot-spot solution to the shell solution.

3.4 Shell Dynamics.

In the initial stage of the deceleration phase, the hot-spot is heated and compressed by the piston action of the shell. If the hot-spot pressure is sufficiently large, a thermal instability is driven by the absorbed fusion power leading to a fast increase of the hot-spot energy. Such an instability is referred to as thermonuclear ignition.

As mentioned in Section 3.1, the deceleration phase starts after the shock reflected from the center of the capsule interacts with the incoming shell. For simplicity, we assume that after the first shock reflection off the shell, all subsequent shocks are weak and do not produce large pressure jumps within the hot-spot or the shell. In this section, we describe a simple model ([42]) for the shell dynamics which captures the essential physics of the implosion and hot-spot compression. We approximate the shell as a thin incompressible spherical layer which acts like a piston on the hot-spot. The shell motion is governed by Newton’s law balancing the shell inertia with the outward force of the hot-spot pressure:

$$M_{shell} \frac{d^2}{dt^2} R_{hs} = 4\pi R_{hs}^2 P_{hs}$$  \hspace{1cm} (3.63)

Observe that, following the thin shell approximation, the trajectory of the shell center of mass is approximately equal to the hot-spot radius $R_{hs}$. The hot-spot energy balance 3.50 can be rewritten as

$$\frac{d}{dt} [P_{hs}^{1/3} R_{hs}^2] = D_a P_{hs}^{\frac{3}{4} \alpha} R_{hs}^3$$  \hspace{1cm} (3.64)

and equations 3.63, 3.64 can be combined into a single ordinary differential equa-
\[
\frac{d}{dt} \left( \frac{d^2 R_{hs}}{dt^2} R_{hs}^3 \right) = K R_{hs} \left( \frac{d^2 R_{hs}}{dt^2} \right)^2,
\]
(3.65)

where \( K = \Gamma D_\alpha M_{shell}/(4\pi) \) and \( \Gamma \) has been set equal to 5/3. The initial conditions for the equation 3.65 are:

\[
\begin{align*}
R_{hs}(0) &= R_0, \quad \dot{R}_{hs}(0) = -V_0, \\
\ddot{R}_{hs}(0) &= \frac{4\pi R_{hs}^2 P_{hs}(0)}{M_{shell}},
\end{align*}
\]
(3.66)

where \( R_0 \) is the average shell radius and \( V_0 \) is the average shell velocity at the beginning of the deceleration phase.

We introduce the dimensionless variables:

\[
\begin{align*}
\hat{R} &= \frac{R_{hs}}{R_0}, \quad \tau = \frac{t}{\tau_0}, \\
\tau_0 &= \left( \frac{M_{shell}}{4\pi P_{hs}(0) R_0} \right)^{1/2}
\end{align*}
\]
(3.67)

(3.68)

The equation 3.65 and initial conditions 3.66 in the non-dimensional form are:

\[
\frac{d}{d\tau} \left( \frac{d^2 \hat{R}}{d\tau^2} \hat{R}^3 \right) = A\hat{R} \left( \frac{d^2 \hat{R}}{d\tau^2} \right)^2,
\]
(3.69)

\[
\hat{R}(0) = 1, \quad \dot{\hat{R}}(0) = -B, \quad \ddot{\hat{R}}(0) = 1
\]
(3.70)

where \( B = \tau_0 V_0 / R_0, \ A = K / (R_0 \tau_0) \).

Equation 3.69 represents a very simple model of the shell dynamics and provides a qualitative understanding of the deceleration phase and the onset of ignition. A more accurate model can be developed by including the compressibility of the shell and the shock propagation within the shell (see [43] (in preparation)).

It is important to notice that as long as the \( \alpha \)-particle heating is smaller than the compression work, the right hand side of the equation 3.69 can be neglected.
and the shell trajectory is given by

$$\hat{R} = \sqrt{1 - 2\tau B + \tau^2(1 + B^2)}$$  \hspace{1cm} (3.71)

leading to the following values of the stagnation radius and maximum deceleration:

$$R_{stag} = \frac{R_0}{\sqrt{1 + B^2}}, \quad g_{max} = \left(1 + B^2\right)^{3/2} \frac{R_0}{\tau_0^2}$$  \hspace{1cm} (3.72)

The hot-spot pressure can be derived from 3.48 leading to

$$P_{hs}(t) = P_{hs}(0) \left(\frac{R_0}{R_{hs}(t)}\right)^5 = \frac{P_{hs}(0)}{(1 - 2\tau B + \tau^2(1 + B^2))^{5/2}}$$  \hspace{1cm} (3.73)

and its stagnation value is

$$P_{stag} = P_{hs}(0) \left(1 + B^2\right)^{5/2}$$  \hspace{1cm} (3.74)

For direct-drive NIF-like capsules, the averaged parameters at the beginning of the deceleration phase are $P_{hs}(0) \approx 900\, Mbar$, $M_{shell} \approx 1\, mg$, $R_0 = R_{hs}(0) = 0.5[R_{inn}(0) + R_{out}(0)] \approx 240\, \mu m$, $V_0 = 0.5(U_{inn}(0) + U_{out}(0)) \approx 380\, \mu m/ns$, leading to

$$R_{stag} \approx 75\, \mu m, \quad g_{max} \approx 2100\, \mu m/ns^2$$  \hspace{1cm} (3.75)

Equations 3.73-3.74 give scaling relations for the stagnation radius and maximum deceleration in terms of hot-spot and shell parameters at the beginning of the deceleration phase. Such relations are valid for very thin shells and as long as the hot-spot is not ignited. If the $\alpha$-particle heating becomes important, the right hand side of the equation 3.69 must be retained. In this case, equation 3.69 yields singular solutions for some values of the dimensionless parameters $A$ and $B$.

A typical singular solution is shown in figure 3.4 representing the trajectories of the hot-spot radius for various initial velocities of the shell and $A = 0.1$. The singular solution means that the shell is ejected outward with an infinite velocity.
Figure 3.4: Trajectories of the hot-spot radius. For a given parameter $A = 0.1$, ignition starts when $B = 2.41$.

Such solutions correspond to the thermal instability of the hot-spot that we denote as "ignition". The singularity is due to the fact that the fusion reaction rate $\langle \sigma v \rangle$ is taken to be proportional to $T^2$ and therefore it diverges as $T \to \infty$. In reality, $\langle \sigma v \rangle$ is bounded and the shell velocity is always finite. However, the singularity in the solution of 3.69 provides a simple and robust definition of the ignition in the thin shell model. We therefore conclude that the hot-spot is ignited if the solution of the equation 3.69 is singular. A straightforward analysis yields the condition for singular solutions in the limits of $B \gg 1$ and $B \ll 1$. If $B \gg 1$, it is possible to rewrite the 3.69 in such a way that the only dimensionless parameter is $AB^3$, thus showing that singular solutions require $AB^3$ to be larger than a critical value. The critical value can be determined numerically, and the ignition condition for $B \gg 1$ is $AB^3 > 1.7$. In the opposite case $B \ll 1$, the relevant dimensionless parameter
is $A$ and the solution is singular for $A > 2.2$. This case, however, is not relevant to ICF implosions as it requires that the deceleration phase starts near the stagnation point.

The parameters $(A, B)$ determines the ignition conditions for the spherical thin shell. The diagram shown in the figure 3.5 represent the ignition domain obtained from numerical solution of equation 3.69 for various $A$ and $B$.

![Ignition domain diagram](image)

Figure 3.5: The ignition domain in coordinates $(A, B)$. No ignition in the shaded part of the diagram.

The physical interpretation of the ignition threshold $AB^3 > 1.7$ is straightforward. We rewrite the hot-spot energy equation 3.50 in the following intuitive form:

$$\frac{1}{E_{hs}} \frac{dE_{hs}}{dt} = \frac{5}{3} D_\alpha P_{hs} - 2 \frac{\dot{R}_{hs}}{R_{hs}}$$

(3.76)

where $E_{hs} = (4\pi/3)P_{hs}R_{hs}^3$ is the hot-spot energy. Equation 3.76 shows that
after stagnation, the second term in the right hand side is negative and represents
the inverse hot-spot decompression time \( \tau_{\text{dec}} = R_{\text{hs}}/(2\dot{R}_{\text{hs}}) \) due to the outward
motion of the shell pushed by the hot-spot pressure. The decompression time can
be estimated from the Newton’s law of the shell motion 3.63 by setting \( \tau_{\text{dec}} = \)
\( 0.5\sqrt{R_{\text{hs}}/\dot{R}_{\text{hs}}} \):

\[
\tau_{\text{dec}} = \frac{1}{2} \sqrt{\frac{M_{\text{shell}}}{4\pi P_{\text{hs}} R_{\text{hs}}}}
\]

(3.77)

The first term in the right hand side of 3.76 represents the inverse \( \alpha \)-particle
heating time \( \tau_\alpha = 1/(\Gamma D_\alpha P_{\text{hs}}) \). Ignition occurs after stagnation if the alpha
heating time is shorter than the decompression time:

\[
\tau_\alpha(\text{stag.}) < \tau_{\text{dec}}(\text{stag.}),
\]

(3.78)

and the ignition process is triggered because the hot-spot pressure starts increasing,
and \( \tau_\alpha \sim 1/P_{\text{hs}} \) becomes shorter. The decompression time \( \tau_{\text{dec}} \sim 1/\sqrt{P_{\text{hs}}} \) decreases
less than the \( \alpha \)-particle heating time, thus leading to a further increase in pressure
and a thermal explosive instability. In order to estimate the ignition threshold, we
use the stagnation values 3.73-3.74 (obtained without \( \alpha \)-particles):

\[
\tau_\alpha(\text{stag.}) \approx \frac{3}{5} \frac{1}{D_\alpha P_{\text{hs}}(0)} \frac{1}{(1 + B^2)^{5/2}} = \frac{\tau_\alpha(0)}{(1 + B^2)^{5/2}}
\]

(3.79)

\[
\tau_{\text{dec}}(\text{stag.}) \approx \frac{1}{2} \left( \frac{R_{\text{stag}}}{f_{\text{stag}}} \right)^{1/2} = \frac{\tau_0}{2(1 + B^2)},
\]

(3.80)

where \( \tau_0 \) (defined in 3.68) represents the decompression time if the shell is stag-
nated at \( t = 0 \) (i.e. \( \tau_{\text{dec}}(0) = 0 \)). Substitution of 3.79,3.80 into 3.78 yields the
approximate ignition condition

\[
\frac{\tau_{\text{dec}}(0)}{\tau_\alpha(0)} (1 + B^2)^{3/2} = A (1 + B^2)^{3/2} > 2
\]

(3.81)

Observe that \((1 + B^2)^{3/2}\) is the amplification factor for \( A \) due to the hot-spot
compression by the shell and the term \( B^2 \) is proportional to the ratio of the shell
kinetic energy and the hot-spot internal energy at the beginning of the deceleration phase:

\[ B^2 = \frac{2}{3} \frac{1}{5} \frac{M_{shell} V_0^2}{P_{hs}(0) R_0^3}, \]  

(3.82)

which is typically much greater than unity. A fit of the numerical results gives the following approximation of the ignition threshold:

\[ A \left[ 1 + \left( \frac{5}{3} \right)^{1/3} B^2 \right]^{3/2} > \sqrt{5} \]  

(3.83)

which is in remarkably good agreement with the result of the simple derivation 3.81.

It is important to emphasize that the thin shell model underestimates the ignition condition because it assumes that the entire shell acts like a piston providing the compression work. A more accurate solution should take into account the effect of the shell compressibility as described in [43] (to be published).

### 3.5 One and Two-Dimensional Simulation of the Deceleration Phase.

#### 3.5.1 Planar Geometry.

Many essential features of the deceleration phase of an ICF capsule are reproduced by a simple planar model consisting of a dense foil compressing a low density plasma ("gas") against a rigid wall. We consider a domain with the "wall" on its left boundary, e.g. the normal component of hydro velocity is zero on the wall, and all the other physical values have zero normal derivatives there (reflective boundary conditions). We assume that the right boundary is unperturbed and all quantities have zero derivatives. A strong shock wave is propagating toward the wall, and a dense cold shell is following the shock compressing the gas ahead of it
Figure 3.6: The shell surface velocity versus time in the planar model. Two stages of the deceleration can be observed: deceleration by a series of shocks \( t < 1.7\text{ns} \) and the continuous deceleration \( t > 1.7\text{ns} \).

(see figure 3.7). In order to simulate NIF-like conditions we consider a domain with a length of 600\( \mu \text{m} \) and a shell thickness of about 60\( \mu \text{m} \). The shell boundaries are smooth with an initial density gradient scale length of about 20 \( \div \) 90\( \mu \text{m} \). Plasma is assumed to follow the ideal gas equation of state.

The initial shock is reflected from the wall and interact with the moving shell, giving rise to a transmitted shock, which compresses the shell, and a reflected shock, propagating off the shell toward the wall. Multiple shocks are reflected off the wall and the shell leading to an impulsive deceleration of the shell. After each reflection the shock weakens and the gas is heated up producing the so called “hot-spot”. As the shocks weaken, the deceleration of the shell becomes continuous, the time history of the shell surface velocity is given in the figure 3.6.

The moment in time when the surface velocity vanishes is referred to as “stag-
Figure 3.7: The density and temperature distributions in the beginning of calculation and at stagnation.

nation”. One dimensional initial and stagnation distributions of the density and temperature are drawn in the figure 3.7.

The Rayleigh-Taylor instability can be reduced by several factors: compression of the shell during the deceleration phase, finite density gradient scale length and mass ablation at the inner shell surface.

According to chapter 6 of [2], a simple estimate of density gradient scale length is \( L_m \sim 0.2R_{hs} \), where \( R_{hs} \) is the hot spot radius. We have performed a series of 1-D simulations varying the spatial resolutions. The simulation results, presented in figure 3.8, show that the density gradient scale length depends on the numerical resolution, and it is well below \( 0.2R_{hs} \approx 2\mu m \) (being \( R_{hs} \approx 10\mu m \)). In each run \( L_m = \min ((d\rho/dx)/\rho)^{-1} \) was determined at the stagnation point. As the resolution increases, \( L_m \) decreases and asymptotically reaches the value \( \approx 0.4\mu m \). We emphasize the importance of the high resolution to correctly resolve the density
Figure 3.8: Density gradient scale length at stagnation obtained from numerical simulations with different resolutions. For the considered model the value of $L_m$ does not change significantly if $\Delta x < 0.05\mu m$.

gradient scale length and conclude that the scale $L_m$ should be resolved by at least 10 grid cells.

The figure 3.9 shows the time history of $L_m$. The density gradient scale length reaches its minimum value in the vicinity of the stagnation point and does not change significantly during the 100$\mu$s interval before stagnation which occurs at $t \approx 1.8\mu s$.

The other important stabilizing mechanism is the mass ablation. A simple estimate of the ablation velocity in planar geometry can be obtained by balancing the heat flux leaving the hot-spot with the internal energy flux of the material ablated off the shell. A short calculation leads to the following form of the ablation velocity:

$$V_a \approx \frac{2}{5} \frac{\kappa (T_{\text{max}}_{\text{hotspot}})}{\rho_{\text{shell}} R_{\text{hotspot}}}$$

(3.84)

The time history of the ablation velocities obtained from numerical simulations
Figure 3.9: The density gradient scale length obtained numerically versus time.

Figure 3.10: Time history of the ablation velocity in planar geometry.
and formula 3.84 are presented in figure 3.10. Both curves indicate that the ablation velocity varies between 2 and 6\(\mu m/ns\). In order to estimate the impact of ablation velocity on the RT instability, we consider a simple formula for the RT growth rate derived for low Froude numbers in [10]. The acceleration of the shell inner surface is deduced from its trajectory \((g = \ddot{R}_h)\). Figure 3.11 shows the time evolution of the acceleration which value at stagnation is \(g \approx 800\mu m/ns^2\).

The Froude number \(Fr = V_a^2/gL_0\) characterizes the ablation front with respect to RT instability. Following [15], \(L_0\) is defined as \(L_0 = L_m\nu/(\nu + 1)^{\nu+1}\) leading to \(L_0 \approx 0.05\mu m\). Figure 3.12 shows the time evolution of the Froude number which is well below unity during the entire deceleration phase.

Using the simple model derived in [10]

\[
\gamma = \sqrt{\frac{kg}{1 + kL_m}} - 1.3kV_a \tag{3.85}
\]

it is easy to estimate the cutoff wave number. Equation 3.85 indicates that ablation induces a cutoff at sufficiently short wavelength. Using \(L_m = 0.4\mu m\),
Figure 3.12: Time history of the Froude number in planar geometry.

\[ V_a = 4 \mu m/ns, \quad g = 800 \mu m/ns^2, \] we find that the unstable spectrum exhibits a cutoff at \( k \approx 7.5 \mu m^{-1} \)

More detailed theoretical calculations of the instability growth have been carried out using the "sharp boundary model" developed in [12]. The model includes additional stabilizing factors such as finite compressibility. According to that model, the evolution of the perturbation \( \eta \) is described by the following ordinary differential equation:

\[ \rho \frac{d}{dt} \left( \frac{1}{\rho} \frac{d}{dt} (\rho \eta) \right) + 4kV_a \rho \frac{d\eta}{dt} + \eta \left[ 2k \frac{d}{dt} (\rho V_a) + k \rho \left( \frac{kV_a^2}{(kL_0)^{1/\nu}} - \frac{g}{1 + kL_m} \right) \right] = 0 \quad (3.86) \]

The first term in 3.86 represents the compressibility effect, the second one represents the ablative stabilization, and the stabilizing effect of the finite density gradient scale length is expressed by the denominator of the last term in the parentheses. Thus, it is possible to study the influence of each stabilizing factor by keeping only one of these terms and neglecting the others. Figure 3.13 compares the inner shell surface distortion obtained using the two-dimensional numerical simulations and
the sharp boundary model taking into account such stabilizing factors (as finite gradient scale length, mass ablation and compressibility) one at the time. The "compressibility" curve represents the solution of the equation 3.86 with $V_a = 0$ and $L_m = 0$ and retaining the effect of compressibility $\rho = \rho(t)$. The "ablation" curve represents the solution of 3.86 with $\rho = \text{const}$, $L_m = 0$ and retaining the effect of ablation. The $L_m$ curve represents the solution of 3.86 with $\rho = \text{const}$, $V_a = 0$ and retaining the effect of finite density gradient scale length. The "prediction" curve represents the solution of the equation 3.86 including all stabilizing effects: compressibility, ablation and finite density gradient scale length. Observe that the compressibility of the shell has a little influence on the instability, finite gradient scale length and mass ablation are of equal importance for this particular wavelength, and the combination of these factors significantly reduces the instability growth rate.

We have compared the growth rates obtained by the numerical simulations, the sharp boundary model and the simple equation 3.1. The theoretical and numerical instability spectra are given in figure 3.14. Though the cutoff is not reached in simulations because of the large computational time required for short wavelength modes simulation, it is shown that the mass ablation plays a key role in reducing the instability growth rate.

3.5.2 Spherical Geometry.

The planar model described in the previous section is a useful tool for the physical understanding of the deceleration phase RT instability. All the relevant stabilizing mechanisms are present in the planar simulations and the RT growth rate is significantly reduced with respect to its classical value.
Figure 3.13: Growth of a single unstable mode with wavelength $\lambda = 10\mu m$. The simulation result is compared with the analytical solutions of equation 3.86. All equations were solved with the initial conditions obtained in the simulation.
Figure 3.14: Theoretical and numerical growth rates. Dots represent the results of numerical simulation, the solid curves are the simple formula 3.1 and the solution of 3.86, the dashed line is the formula 3.85. Numerical simulations confirm the theory predicting significant reduction of RT instability by the mass ablation.

Though qualitatively similar to the planar case, the evolution of the deceleration phase RT instability in a spherical ICF capsule could be significantly different because of the higher pressures and temperatures achieved in a spherical implosion. Higher pressures and temperatures lead to a larger deceleration and ablation velocity respectively leading to different growth rates with respect to the planar case.

Our analysis focuses on a NIF-like capsule with a $345 \mu m$ thick shell of DT ice with an inner radius of $1350 \mu m$ driven by a 9.3ns, 1.5 MJ laser pulse which sets the shell on a $\alpha = 3$ adiabat. The shell is filled with DT gas with a density of $2 \cdot 10^{-4} \text{ gr/cm}^3$ and a temperature of 18 K.

We have used the 1D-code LILAC output at 9.5 ns characterizing the beginning
of the coasting phase and used it as the input for a high resolution 2-D Eulerian hydro-code solving the single-fluid mass, momentum and energy equations including Spitzer conduction, local alpha deposition and bremsstrahlung on a very fine grid. Aside from the bremsstrahlung losses, the code solves the same single fluid equations on which the theory is based providing a robust check of the theoretical results described in Sections 3.2-3.4. The high resolution is needed to correctly simulate the growth of short wavelength modes. The R-T evolution is investigated by introducing a small amplitude 2-D perturbation of the hydrodynamic variables at about 200ps before stagnation when the continuous deceleration phase begins.

All the numerical results presented here have been obtained using the Eulerian hydro-code described in Chapter 4. We emphasize that both theory and simulations do not include the full ignition and burn wave propagation physics (radiation transport and alpha particle diffusion) and their result may not apply at the time of the burn wave propagation occurring after stagnation.

The behaviour of the spherical capsule is qualitatively similar to that of the planar foil. The reflected shocks decelerate the shell impulsively while the increasing hot-spot pressure leads to a continuous deceleration. The DPRT develops during the continuous deceleration phase when the growth is approximately exponential.

The density gradient scale length $L_m \approx 1\mu m$ is bigger than in planar case, but still less than $0.2R_{\text{hotspot}}$.

As shown in figure 3.15, the typical inner surface acceleration at stagnation ($t = 0$) is about $3000 - 4000\mu m/ns^2$.

Similarly to the planar case, we expect that finite ablation velocity and density gradient scale length will significantly reduce the RT growth rate.

The ablation velocity is calculated both numerically and analytically. Since it is not clear where on the shell inner surface ablation takes place, the numerical
Figure 3.15: The inner surface acceleration versus time.

The definition of ablation velocity relies upon the assumption that the ablation front advances at the location of the peak density within the compressed shell. In the limit of a sharp inner surface and using the conservation of mass flow through a sharp discontinuity, the ablation velocity can be written in the following form

$$V_a = (V(r_\omega) - \dot{R}_{hs})\omega$$  \(3.87\)

where \(V(r_\omega)\) is the hydro velocity calculated at \(r_\omega\) and \(r_\omega\) is the radial location on the shell inner surface (equal to the hot-spot radius \(R_{hs}\)) where the density is \(\omega\) times the peak shell density. Equation 3.87 is suitable for a numerical estimate of the ablation velocity. However, it is important to emphasize, that the equation 3.87 underestimates the ablation velocity at the beginning of the deceleration phase when the heat front is penetrating through the shell at a density level well below the peak density.

The figure 3.16 represents the time evolution of \(V_a\) near stagnation calculated from the simulations using the equation 3.87 and the theoretical formula 3.57.
The good agreement between theory and simulation is a further confirmation of the validity of the approximations used in the theoretical derivation.

The density gradient scale length can be easily determined numerically by calculating the minimum of \( \rho d \sigma/d \rho \). Figure 3.17 shows the time evolution of the density gradient scale obtained from the numerical results and the theoretical formula 3.62. Both numerical and analytic results seem to confirm that the density gradient scale length of NIF-like capsule is well below other estimates reported in the literature (see [2]).

The growth of large \( l \) modes can be determined using the planar results of [10, 12] derived for the acceleration-phase RT instability. For a NIF-like capsule during the continuous deceleration-phase before stagnation \( g \approx 3100 \, \mu m/ns^2 \), \( V_a \approx 17 \, \mu m/ns \), \( L_m \approx 1.5 \, \mu m \), \( R_{hs} \approx 50 - 70 \, \mu m \) leading to a Froude number \( Fr \approx 0.5 \) where \( Fr = V_a^2/gL_0 \) and \( L_0 = 0.12L_m \). Using equation (23) and figure 6
Figure 3.17: The density gradient scale length versus time in spherical case. Stagnation time is at \( t = 0 \).

of [10], we determine the appropriate growth rate formula

\[
\gamma = 0.9\sqrt{\frac{kg}{1 + kL_m}} - 1.4kV_a, \tag{3.88}
\]

where \( k \sim l/R_{hs} \) for large \( l \)'s.

The linear DPRT growth rates have been also investigated numerically using 2D simulations of the deceleration phase for the NIF capsule described earlier. A single mode perturbation is a Legendre polynomial \( P_l(\theta) \), which can be approximated as \( \cos(l\theta) \) for \( l \gg 1 \). The spectrum of the instability is given in figure 3.18. The numerical simulation results (dots) are compared with the analytical formula 3.88 (solid line). The dashed line represents the classical result 3.1. Except for \( l = 2, 4 \), the numerical growth rates are calculated in the 100 ps time interval before stagnation. The modes \( l = 2, 4 \) grow exponentially only after the shell is stagnated, and their growth rates are calculated during 50 ps after stagnation.

It is important to observe that the simulation results agree with the planar
Figure 3.18: The deceleration phase RT growth rates in the spherical case plotted versus the Legendre mode number. The rates are evaluated at \( \sim 100\text{ps} \) time range before stagnation. The spectrum is strongly influenced by the mass ablation, and has the cutoff at \( l \approx 90 \). The dashed line is the classical RT result 3.1.

theory only for \( l > 20 \). Low \( l \) modes grow faster than predicted by 3.88, indicating that convergence effects may reduce the ablative stabilization for long wavelength modes. The cutoff in the deceleration phase RT spectrum occurs at approximately \( l \approx 90 \) and it is caused by mass ablation. We conclude that modes with \( l > 90 \) do not contribute to mixing of the shell and hot-spot plasmas during the deceleration phase.
Chapter 4

Codes.

4.1 Lagrangian Code.

4.1.1 Motivation.

The analytic theory of long wavelength feedout described in Sections 2.2-2.4 has been developed for long wavelength modes satisfying the condition $kd < 1$. To verify the theory, determine the scope of its applicability and investigate the feed-out of short wavelengths ($kd > 1$), a two-dimensional numerical code has been developed to solve the conservation equations described in Section 2.2. The code simulates the one and two-dimensional evolution of a laser accelerated planar foil.

Since the accelerated foil travels over a distance several times its thickness, it is natural to use a Lagrangian, or "material", description of the fluid. This requires labeling each fluid particle and following it during the calculation.

The laser absorption is included through a heat source localized at the critical surface where the plasma frequency equals the laser frequency. Alternatively, the pressure instead of the heat source can be specified at the foil outer surface. This
option has been used in the simulation of the long wavelength feedout as heat conduction does not significantly affect the evolution of long wavelength perturbations. Instead, the growth of short wavelength modes is substantially reduced by the finite heat conductivity and thermal transport must be retained in the simulations. Following [11], the effects of radiation transport are included through a modified thermal conductivity $\kappa = \kappa T^\nu$, where $\nu$ depends on the material atomic number $Z$. In CH plasmas with $Z = 3.5$, a significant fraction of the energy transport occurs through radiation and $\nu$ is approximately equal unity. For DT, $Z = 1$, electron heat conduction is the dominant energy transfer mechanism leading to $\nu = 2.5$.

4.1.2 Coordinate Systems.

The terms $(x, y)$ and $(\xi, \eta)$ denote the Eulerian and Lagrangian coordinate systems respectively. Given the transformation $x = x(\xi, \eta, t), y = y(\xi, \eta, t)$, one can easily rewrite the system 2.2-2.4 in terms of $(\xi, \eta, t)$. It should be mentioned, that the only conditions that must be satisfied for the Lagrangian system are:

$$\frac{\partial x}{\partial t} \bigg|_{\xi, \eta=const} = v_x, \quad \frac{\partial y}{\partial t} \bigg|_{\xi, \eta=const} = v_y, \quad (4.1)$$

where $v_x$ and $v_y$ are the components of the fluid hydrodynamic velocity. It is not necessary for the Lagrangian coordinates $(\xi, \eta)$ to be the initial position of the fluid elements or to have equal mass for all fluid particles. In order to simplify the finite differencing, we choose a rectangular Lagrangian grid.

In the Lagrangian frame of reference the conservation equations can be written in the following form:

1. Conservation of mass

$$\rho J = \text{const} \quad (4.2)$$
Figure 4.1: Eulerian \((x, y)\) and Lagrangian \((\xi, \eta)\) grids. The domain in the \((x, y)\) space is distorted, and the boundary conditions have to be assigned on the distorted moving boundaries.

2. Conservation of momentum

\[
\rho J \frac{\partial}{\partial t} u_x = \frac{\partial y}{\partial \xi} \frac{\partial P}{\partial \eta} - \frac{\partial y}{\partial \eta} \frac{\partial P}{\partial \xi} \tag{4.3}
\]

\[
\rho J \frac{\partial}{\partial t} u_y = - \frac{\partial x}{\partial \xi} \frac{\partial P}{\partial \eta} + \frac{\partial x}{\partial \eta} \frac{\partial P}{\partial \xi} \tag{4.4}
\]

3. Conservation of energy

\[
\rho c_v \frac{\partial}{\partial t} T = -\rho P \frac{\partial}{\partial t} \left[ \frac{1}{\rho} \right] + S + \frac{1}{J} \left\{ \left( \frac{\partial y}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial}{\partial \eta} \right) \frac{\kappa}{J} \left( \frac{\partial y}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial}{\partial \eta} \right) + \left( \frac{\partial x}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial x}{\partial \xi} \frac{\partial}{\partial \eta} \right) \frac{\kappa}{J} \left( \frac{\partial x}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial x}{\partial \xi} \frac{\partial}{\partial \eta} \right) \right\} T, \tag{4.5}
\]

where \(\kappa\) is the heat transfer coefficient, \(T\) is the temperature, \(\overline{q} = -\kappa \nabla T\), \(J\) is the
jacobian of the coordinate system transformation:

\[ J = \frac{\partial (x, y)}{\partial (\xi, \eta)} = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi}. \]  

\[ (4.6) \]

In the absence of heat transfer and sources, we have for the energy conservation (using the ideal gas equation of state 2.5):

\[ \frac{1}{\Gamma - 1} \frac{\partial}{\partial t} \left( \frac{P}{\rho} \right) - \frac{\partial P}{\partial t} \left( \frac{1}{\rho} \right) = 0 \]  

\[ (4.7) \]

Additionally, the two extra equations 4.1 must be solved to follow the fluid elements according to the Lagrangian description of the flow.

The domain of the simulation extends between 0 and \( L_\xi \) in \( \xi \)-direction and between 0 and \( L_\eta \) in \( \eta \)-direction:

\[ 0 < \xi < L_\xi, \quad 0 < \eta < L_\eta \]  

\[ (4.8) \]

where \( L_\xi \) and \( L_\eta \) are arbitrary constants.

The periodic boundary conditions are applied in the \( \xi \)-direction:

\[ f(0, \eta) = f(L_\xi, \eta), \]  

\[ (4.9) \]

where \( f(\xi, \eta) \) represents all physical values except for the trajectory component \( x(\xi, \eta) \) for which \( x(0, \eta) + L_x = x(L_\xi, \eta) \), where \( L_x \) is the \( x \)-dimension of the domain in the physical space \( (x, y) \). The Dirichlet boundary conditions

\[ P(t, \xi, 0^-) = P^{ext}(t, \xi), \quad P(t, \xi, L_\eta^+) = P^0(t, \xi) \]  

\[ (4.10) \]

are applied when heat transfer is not included in the system 4.1-4.4, 4.7 and the external pressure \( P^{ext}(t, \xi) \) is assigned at the foil front surface.

If heat transfer and energy sources are included, the Neumann boundary conditions

\[ \frac{\partial T}{\partial n} \bigg|_{\eta=0} = \frac{\partial T}{\partial n} \bigg|_{\eta=L_\eta} = 0 \]  

\[ (4.11) \]
are used at the outer surface of the low density plasma ablated off the foil. Here $\vec{n}$ denotes the normal direction to the surface. The conditions 4.11 imply that the heat flux vanishes at the boundaries $\eta = 0, \eta = L_\eta$.

### 4.1.3 Artificial Terms.

In order to treat shock propagation problems correctly, an artificial dissipative mechanism is introduced to resolve the shock front. The Von Neumann-Richtmyer concept of artificial terms was originally formulated in one spatial dimension to describe the propagation of shock waves (see [44, 45]). It is shown in [44], that the substitution $P \rightarrow P + Q$ in the right hand sides of the conservation equations does not violate the Rankine-Hugoniot conditions for shock solutions in the regions where $Q = 0$ (far enough from the discontinuity). Thus the properly chosen artificial term $Q$ may smooth the density profile over several points, without altering the Rankine-Hugoniot relations.

![Figure 4.2: The density profile obtained in the simulation where the artificial terms were used.](image-url)
The standard one-dimensional form of the Von Neumann viscosity $Q$ is:

$$Q = a_0 \rho (\Delta u)^2 \quad \text{if} \quad \Delta u < 0,$$

$$Q = 0 \quad \text{if} \quad \Delta u \geq 0$$

where $\rho$ is the fluid density, $\Delta u$ is the velocity change across a cell, $a_0$ is the adjustable constant (usually $a_0 \approx 4$).

A generalization of the method for two spatial dimensions was developed in [45] and is used in this code. The standard requirements to the artificial viscosity terms are met: shocks manifest themselves as approximate discontinuities in $\rho$, $P$ and $T$, no internal boundary conditions are required by the method, and the basic conservation laws expressed by the Rankine-Hugoniot conditions are satisfied.

![Diagram](image)

Figure 4.3: Calculation of characteristic grid length. $l$ is the line in the direction of acceleration. The zone center $c$ is taken from average of the corner coordinates.

The artificial viscosity term used in the code has the following form:

$$Q = a_0 \rho L^2 \left(\frac{ds}{dt}\right)^2 \quad \text{if} \quad \frac{ds}{dt} < 0,$$
\[ Q = 0 \quad \text{if} \quad \frac{ds}{dt} \geq 0, \]

where \( \rho \) is the zone density, \( L \) is the characteristic grid length:

\[ L = \frac{2A}{d_1 + d_2 + d_3 + d_4}, \quad (4.14) \]

\( A \) is the zone area, \( d_i \) is the perpendicular distance from the grid node \( i \) to a line through the grid center in direction of acceleration,

\[ \frac{ds}{dt} = \frac{\partial \tilde{x}}{\partial x} \cos^2 \alpha + \frac{\partial \tilde{y}}{\partial y} \sin^2 \alpha + \left( \frac{\partial \tilde{x}}{\partial x} + \frac{\partial \tilde{y}}{\partial y} \right) \cos \alpha \sin \alpha, \quad (4.15) \]

the angle \( \alpha \) is the direction of acceleration (see figure 4.3). The spatial derivatives in 4.15 are calculated according to the general rules of derivative transformation:

\[ \frac{\partial f}{\partial x}_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{J} \left[ \frac{(y_{i+1,j+1} - y_{i+1,j} + y_{i,j+1} - y_{i,j}) (f_{i+1,j+1} - f_{i,j+1} + f_{i+1,j} - f_{i,j})}{4\Delta \xi \Delta \eta} \right. \]

\[ - \frac{(y_{i+1,j+1} - y_{i,j+1} + y_{i+1,j} - y_{i,j}) (f_{i+1,j+1} - f_{i+1,j} + f_{i,j+1} - f_{i,j})}{4\Delta \xi \Delta \eta} \right] \quad (4.16) \]

\[ \frac{\partial f}{\partial y}_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{J} \left[ \frac{(x_{i+1,j+1} - x_{i,j+1} + x_{i+1,j} - x_{i,j}) (f_{i+1,j+1} - f_{i+1,j} + f_{i,j+1} - f_{i,j})}{4\Delta \xi \Delta \eta} \right. \]

\[ - \frac{(x_{i+1,j+1} - x_{i+1,j} + x_{i,j+1} - x_{i,j}) (f_{i+1,j+1} - f_{i,j+1} + f_{i+1,j} - f_{i,j})}{4\Delta \xi \Delta \eta} \right] \quad (4.17) \]

The modified equations including the above artificial mechanism are easily obtained from the equations 4.2-4.5 by the substitution \( P \to P + Q \) in the right-hand side.

### 4.1.4 Spatial Staggering.

All variables are either cell-centered or corner-centered in the code. The center of each cell is defined in the Lagrangian coordinate system \((\xi, \eta)\). The spatial staggering is standard: pressure, density and temperature are the cell-centered values, while the trajectories \( x = x(\xi, \eta, t) \), \( y = y(\xi, \eta, t) \) and velocities \( u_x, u_y \) are sought at the vertices of the grid.
Figure 4.4: Location of physical values on the grid. The fluid element trajectories and velocities are defined at the cell corners, all other values are cell-centered.

4.1.5 Difference Approximation.

The planar Lagrangian equations are approximated on the rectangular grid \((\xi_i, \eta_j)\) (see figure 4.5) where

\[
\begin{align*}
\xi_i &= i \Delta \xi, \quad i = 0, \ldots, N_\xi, \quad \Delta \xi = \frac{L_\xi}{N_\xi} \quad (4.18) \\
\eta_j &= j \Delta \eta, \quad j = 0, \ldots, N_\eta, \quad \Delta \eta = \frac{L_\eta}{N_\eta} \quad (4.19) \\
f_{ij} &= f(\xi_i, \eta_j), \quad (4.20)
\end{align*}
\]

Note that the grid is rectangular only in the Lagrangian coordinates \((\xi, \eta)\), while it can be distorted to any extent in the physical space \((x, y)\).

The following "ghost" points (and "ghost" cells respectively) are introduced to ensure the correct treatment of the boundary conditions:

\[
i = -1 \text{ and } i = N_\xi + 1, \quad j = 0, \ldots, N_\eta \quad (4.21)
\]
Figure 4.5: Rectangular numerical grid in $(\xi, \eta)$ coordinates. The ghost cells (dashed) are introduced in order to approximate the boundary conditions.

are used at the left and right boundaries to determine the periodic boundary conditions, and

$$ j = -1 \text{ and } j = N_\eta + 1, \quad i = 0, \ldots, N_\xi $$

are used at the front and back surfaces to approximate the boundary conditions 4.10 or 4.11. Since the boundary conditions 4.11 contain derivatives in the directions normal to the shell surface, the ghost cells 4.22 are built as shown in figure 4.6. The distance of the ghost points from the boundary can be either assigned explicitly or calculated implicitly using the equation 4.2. In the latter case, the density is found from the equation of state of the adjacent cell. The results proved to not be sensitive to the method of building ghost cells, and so either ways can be used.

The leap-frog scheme with the centered spatial differences is used for the momentum and trajectories equation. The time staggering is standard: the velocities are determined at half-integer time steps, and all other values are at integer time steps, as it is seen in the figure 4.7.
Figure 4.6: The “ghost” cell $BCHG$. $EG \perp AC, FH \perp BD$. $ABCD$ is the boundary.

The components of the Lagrangian particle acceleration $\ddot{a}$ are

\[
a_x = \frac{1}{\rho \hat{J}} \left( \frac{\partial y}{\partial \xi} \frac{\partial (P + Q)}{\partial \eta} - \frac{\partial y}{\partial \eta} \frac{\partial (P + Q)}{\partial \xi} \right)
= \frac{1}{\rho \hat{J}} \left[ \frac{\partial}{\partial \eta} \left( \frac{\partial y}{\partial \xi} (P + Q) \right) - \frac{\partial}{\partial \xi} \left( \frac{\partial y}{\partial \eta} (P + Q) \right) \right]
\]

(4.23)

\[
a_y = \frac{1}{\rho \hat{J}} \left( \frac{\partial x}{\partial \eta} \frac{\partial (P + Q)}{\partial \xi} - \frac{\partial x}{\partial \xi} \frac{\partial (P + Q)}{\partial \eta} \right)
= \frac{1}{\rho \hat{J}} \left[ \frac{\partial}{\partial \xi} \left( \frac{\partial x}{\partial \eta} (P + Q) \right) - \frac{\partial}{\partial \eta} \left( \frac{\partial x}{\partial \xi} (P + Q) \right) \right],
\]

(4.24)

where $Q$ is the artificial term 4.13, and the difference approximation of these equations can be written in the following form:

\[
a_{xij} = \frac{1}{\rho_{ij} \hat{J}_{ij}} \left[ \frac{1}{2\Delta \xi} \left( (y_\xi)_{i+\frac{1}{2},j+\frac{1}{2}} \hat{P}_{i+\frac{1}{2},j+\frac{1}{2}} - (y_\xi)_{i+\frac{1}{2},j-\frac{1}{2}} \hat{P}_{i+\frac{1}{2},j-\frac{1}{2}} + \\
(y_\xi)_{i-\frac{1}{2},j+\frac{1}{2}} \hat{P}_{i-\frac{1}{2},j+\frac{1}{2}} - (y_\xi)_{i-\frac{1}{2},j-\frac{1}{2}} \hat{P}_{i-\frac{1}{2},j-\frac{1}{2}} \right) - \\
\frac{1}{2\Delta \xi} \left( (y_\eta)_{i+\frac{1}{2},j+\frac{1}{2}} \hat{P}_{i+\frac{1}{2},j+\frac{1}{2}} - (y_\eta)_{i-\frac{1}{2},j+\frac{1}{2}} \hat{P}_{i-\frac{1}{2},j+\frac{1}{2}} + \\
(y_\eta)_{i+\frac{1}{2},j-\frac{1}{2}} \hat{P}_{i+\frac{1}{2},j-\frac{1}{2}} - (y_\eta)_{i-\frac{1}{2},j-\frac{1}{2}} \hat{P}_{i-\frac{1}{2},j-\frac{1}{2}} \right) \right]
\]

(4.25)
Figure 4.7: The time stepping scheme where n is the time step index. Once everything is known at \( t = t^n \) and before, one can calculate the acceleration of the fluid element at \( t = t^n \), and therefore its velocity at \( t = t^{n+1/2} \). Then the new position of the element and its density at \( t = t^{n+1} \) can be found, and at last the new pressure \( P^{n+1} \) and temperature \( T^{n+1} \) are obtained from the energy equation.

\[
a_{ij} = \frac{1}{\rho_{ij} J_{ij}} \left[ \frac{1}{2\Delta \xi} \left( (x_{\eta})_{i+1/2,j+1} \hat{P}_{i+1/2,j+1} - (x_{\eta})_{i-1/2,j+1} \hat{P}_{i-1/2,j+1} \right) - \frac{1}{2\Delta \eta} \left( (x_{\xi})_{i+1/2,j+1} \hat{P}_{i+1/2,j+1} - (x_{\xi})_{i+1/2,j-1} \hat{P}_{i+1/2,j-1} \right) \right]
\]

(4.26)

where \( \hat{P}_{i+1/2,j+1} = P_{i+1/2,j+1} + \frac{1}{2 \Delta \xi} (x_{i+1,j+1} - x_{i,j+1} + x_{i+1,j} - x_{i,j}) \) and

\[
(x_{\xi})_{i+1/2,j+1} = \frac{1}{2\Delta \xi} (x_{i+1,j+1} - x_{i,j+1} + x_{i+1,j} - x_{i,j})
\]

(4.27)

\[
(x_{\eta})_{i+1/2,j+1} = \frac{1}{2\Delta \eta} (x_{i+1,j+1} - x_{i,j+1} + x_{i,j+1} - x_{i,j})
\]

(4.28)

\[
(y_{\xi})_{i+1/2,j+1} = \frac{1}{2\Delta \xi} (y_{i+1,j+1} - y_{i,j+1} + y_{i+1,j} - y_{i,j})
\]

(4.29)

\[
(y_{\eta})_{i+1/2,j+1} = \frac{1}{2\Delta \eta} (y_{i+1,j+1} - x_{i+1,j} + x_{i,j+1} - x_{i,j})
\]

(4.30)
The time step super-index \( n \) is assumed in all terms of 4.25-4.30 except for \( Q = Q^{n-1/2} \), because its calculation requires the knowledge of the velocity, which is only available at \( t = t^{n-1/2} \). The artificial term \( Q \) makes the scheme uncentered in time, but it does not seem to be an important disadvantage because \( Q \) does not represent any real physical value. In addition, its effect is dissipative and the solution should not be reversible.

At the periodic boundaries the equations are modified by the substitution \( f_{-1,i,j} \rightarrow f_{N_\xi,j} \) at \( \xi = 0 \), and \( f_{N_\xi+1,i,j} \rightarrow f_{0,j} \) at \( \xi = L_\xi \). If the boundary condition 4.10 is imposed, then \( P_{i+1/2,-1/2} \rightarrow P_{i+1/2}^{\text{ext}} \), and \( P_{i+1/2,N_\eta+1/2} \rightarrow P_{i+1/2}^0 \) at the front and back surfaces respectively.

The new velocity \( \bar{v}^{n+1/2} \) is determined according to the leap-frog scheme:

\[
\begin{align*}
v_{x}^{n+\frac{1}{2}} &= v_{x}^{n-\frac{1}{2}} + a_{x}^{n} \Delta t^{\frac{1}{2}} \\
v_{y}^{n+\frac{1}{2}} &= v_{y}^{n-\frac{1}{2}} + a_{y}^{n} \Delta t^{\frac{1}{2}}
\end{align*}
\tag{4.31}
\tag{4.32}
\]

where the spatial sub-indexes \( i, j \) are omitted and

\[
\Delta t = \frac{\Delta t^{n-\frac{1}{2}} + \Delta t^{n+\frac{1}{2}}}{2}.
\tag{4.33}
\]

The trajectory integration scheme is the standard leap-frog differencing:

\[
\begin{align*}
\frac{x_{ij}^{n+1} - x_{ij}^{n}}{\Delta t^{n+1/2}} &= v_{xij}^{n+1/2}, & \frac{y_{ij}^{n+1} - y_{ij}^{n}}{\Delta t^{n+1/2}} &= v_{yij}^{n+1/2}
\end{align*}
\tag{4.34}
\]

Once the new vertex coordinates are obtained, the jacobian can be found according to 4.6:

\[
\begin{align*}
J_{i+\frac{1}{2}} &= \frac{x_{i+1,j+1} - x_{i,j+1} + x_{i+1,j} - x_{i,j}}{2\Delta \xi} \frac{y_{i+1,j+1} + y_{i,j+1} - y_{i+1,j} - y_{i,j}}{2\Delta \eta} \\
&\quad - \frac{x_{i+1,j+1} + x_{i,j+1} - x_{i+1,j} - x_{i,j}}{2\Delta \eta} \frac{y_{i+1,j+1} + y_{i,j+1} - y_{i+1,j} - y_{i,j}}{2\Delta \xi}
\end{align*}
\tag{4.35}
\]

and the new density is calculated using 4.2.
For numerical stability, an upper bound on the time step is required when the hydrodynamic equations are solved by the explicit method. The Courant-like stability criterion is used:

\[ \Delta t \leq \frac{\Delta x}{c_s}, \]  

(4.36)

where \( \Delta x \) is the cell width and \( c_s \) is the local sound speed. \( \Delta x \) is defined as the minimum distance from the point under consideration to the line connecting the neighbor points (see figure 4.8).

![Figure 4.8: Evaluation of the cell width \( \Delta x \) used to determine the current time step.](image)

This definition of the cell width also prevents the cell from developing a negative volume, at least in subsonic regimes.

### 4.1.6 Solution of the Energy Equation on the Lagrangian Grid.

In the absence of heat transfer and energy sources a modified energy equation is used to describe the energy dynamics:

\[ \frac{1}{\Gamma - 1} \frac{\partial}{\partial t} \left( \frac{P}{\rho} \right) = -(P + Q) \frac{\partial}{\partial t} \left( \frac{1}{\rho} \right) \]  

(4.37)
Taking into account the Lagrangian mass conservation equation 4.2, one can rewrite the equation 4.37 in the following form:

\[
\frac{1}{\Gamma - 1} \frac{\partial}{\partial t} (PJ) = -(P + Q) \frac{\partial}{\partial t} (J),
\]

which can be approximated by standard finite differencing as

\[
P^{n+1}J^{n+1} - P^nJ^n = -(\Gamma - 1) \left( \frac{P^{n+1} + P^n}{2} + Q^{n+\frac{1}{2}} \right) (J^{n+1} - J^n)
\]

(4.39)

Once the jacobian \(J^{n+1}\) is known, the equation 4.39 yields the pressure at the new time step \(P^{n+1}\).

When the heat transfer and sources are included in the model, the equation 4.5 modified by the term \(Q\) is solved. Substituting \(P \rightarrow P + Q\) in the right hand side and using the mass conservation 4.2 and equation of state 2.5, we have

\[
\rho_0J_0c_v \frac{\partial T}{\partial t} + \frac{\rho_0J_0}{J} c_v (\Gamma - 1)T \frac{\partial J}{\partial t} = -Q \frac{\partial J}{\partial t} + SJ + \hat{D}T,
\]

(4.40)

where \(\rho_0\) and \(J_0\) are the initial density and jacobian of the cell, \(J\) is given by 4.6 and \(\hat{D}\) is the operator

\[
\hat{D}T = \left\{ \left( \frac{\partial y}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial}{\partial \eta} \right) \frac{\kappa}{J} \left( \frac{\partial y}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial}{\partial \eta} \right) + \left( \frac{\partial x}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial x}{\partial \xi} \frac{\partial}{\partial \eta} \right) \frac{\kappa}{J} \left( \frac{\partial x}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial x}{\partial \xi} \frac{\partial}{\partial \eta} \right) \right\} T
\]

(4.41)

The following finite difference approximation of the energy equation 4.40 is used:

\[
\frac{\rho_0J_0c_v}{\Delta t} \frac{T^{n+1} - T^n}{\Delta t} + \frac{\rho_0J_0}{J^n + J^{n+1}} c_v (\Gamma - 1) \left( T^n + T^{n+1} \right) \frac{J^{n+1} - J^n}{\Delta t} =
\]

\[
-\frac{Q^{n+\frac{1}{2}} J^{n+1} - J^n}{\Delta t} + \frac{S^n J^n + S^{n+1} J^{n+1}}{2} + \left( \hat{D}T \right)^{n+1}
\]

(4.42)

The implicit scheme 4.42 of the energy equation is used because the stability condition for most explicit schemes \(\Delta t/\Delta x^2 < 1\) is too restrictive on the time step.
size. The exception is the unconditionally stable explicit DuFort-Frankel method (see [46]), which loses its consistency with the original differential equation as the restriction is violated, leading to unphysical results.

The heat transfer term is approximated at $t = t^{n+1}$, thus making the scheme fully implicit. Another choice is the Crank-Nicolson scheme with the combination $(\dot{T}^{n+1} + \dot{T}^n)/2$ used instead of $\dot{T}^{n+1}$. The tests show virtually identical behavior of the simulated system under both approximations. It is necessary to observe that both approaches yield the same first order approximation and so the simpler scheme is preferable. Another argument in favor of 4.42 is its monotonicity that makes it more reliable in nonlinear regimes when rigorous stability analysis is extremely difficult or impossible.

Details of the spatial difference approximation of 4.41 are given in Appendix 4.3.1.

The implicit equation 4.42 leads to a linear algebraic system. Though the energy equation is nonlinear due to the nonlinear heat transfer coefficient $\kappa$, we neglect its variation over a time step and use it explicitly: $\kappa = \kappa^n$. The matrix corresponding to the system has a block structure. The ordering used in the code makes each block tridiagonal, and the full matrix can be either block-tridiagonal or block-tridiagonal with extra corner blocks, depending on what boundary conditions are applied in $\xi$-direction. The corner blocks appear in the most general case of periodic boundary conditions.

The matrix consisting of tridiagonal blocks (with or without corners) appears when a nine-point approximation of the diffusion operator in the energy equation 4.40 is used. The nine-point approximation is of the second order in the Lagrangian space $(\xi, \eta)$ and obviously more accurate than possible five-points approximations. It is shown in [47] that the second order five-point approximation is possible only
Figure 4.9: The matrix corresponding to the set of linear equations approximating the differential equation 4.40 has a block structure. Each block is tridiagonal, and three block diagonals are non-zero. Imposing the periodic boundary conditions adds two extra corner blocks to the matrix.

on rectangular grids, and the attempt to build the finite difference scheme using only five points leads to an inconsistent set of linear equations.

Methods of solution of linear systems appearing in heat and radiation transport problems have been widely studied (see, for example, [48, 49, 50]). The linear system

$$A \bar{x} = \bar{r}$$  \hspace{1cm} (4.43)

appearing in the finite difference approximation of the energy equation is solved iteratively. Here $A$ is the square $N \times N$ matrix with the structure shown in figure 4.9, $\bar{r}$ is the right hand side vector of the system, and $\bar{x}$ is the solution to be found.

The general iteration procedure is explained below. Both sides of the matrix equation 4.43 are multiplied by the matrix $B^{-1}$:

$$B^{-1}A \bar{x} = B^{-1}\bar{r},$$  \hspace{1cm} (4.44)
where $B$ is referred to as the preconditioner. Then 4.44 is rewritten in the following form:

$$\left( I - I + B^{-1}A \right) \vec{x} = B^{-1} \vec{r},$$

(4.45)

where $I$ is the identity matrix, leading to

$$\vec{x} = \vec{x} + B^{-1} \left( \vec{r} - A \vec{x} \right)$$

(4.46)

The equation 4.46 is identical to 4.43. The iterations then are built up as

$$\vec{x}^{(n+1)} = \vec{x}^{(n)} + B^{-1} \left( \vec{r} - A \vec{x}^{(n)} \right),$$

(4.47)

where the super-script denotes the iteration number.

The preconditioner $B$ can be chosen in various ways, and its optimal choice is a challenging problem. One possible approaches is the successive over-relaxation (SOR). Formally it can be represented as

$$\vec{x}^{(n+1)} = \vec{x}^{(n)} + S^{-1} \left( \vec{r} - A \vec{x}^{(n)} \right),$$

(4.48)

$$S = \omega^{-1} \left( D + \omega L \right)$$

(4.49)

where the original matrix $A$ is decomposed into the sum of the diagonal $D$, lower triangular $L$ and upper triangular $U$ matrices:

$$A = D + L + U$$

(4.50)

The relaxation parameter $\omega$ is a real positive number between 0 and 2, the bounds being required for convergence. Usually $\omega > 1$, but in some cases (typically for solutions oscillating in time) $\omega < 1$, and the method is referred to as “under-relaxation”. Note that $\omega = 1$ reduces the SOR method to Gauss-Seidel iterations.

It is important to choose the relaxation parameter $\omega$ properly as it has a strong influence on the rate of convergence. The rigorous calculation of the optimal value
Figure 4.10: The matrix $A$ is represented as a sum of diagonal, lower triangular and upper triangular matrices.

of $\omega$ for the complex nonlinear diffusion problem on a distorted grid is extremely hard, and a simple estimation is performed by monitoring the number of iterations from the last several time steps and extrapolating $\omega$ to the next step. The procedure is continuously repeated in the code to maintain the optimal convergence rate. Typically $\omega$ lies in the range $0.9 < \omega < 1.7$.

The SOR procedure can be represented in the point-wise form. We denote with $a_{i,j}$ the element $i, j$ of the matrix $A$ and rewrite the iterations 4.48 in the following form:

$$x_i^{(n+1)} = (1 - \omega)x_i^{(n)} + \frac{\omega}{a_{i,i}} \left[ \tau_i - \sum_{j=1}^{i-1} a_{i,j}x_j^{(n+1)} - \sum_{j=i+1}^{N} a_{i,j}x_j^{(n)} \right]$$  \hspace{1cm} (4.51)

Some acceleration of convergence of the iterative procedure is supposed to be obtained by ordering the points in a special way. The solution of the considered physical problem has demonstrated that the result is almost independent on the order of points in 4.51.

Another possible algorithm is a block SOR method, also called "successive-line-
over-relaxation" (see [51, 52]). It can be represented in the form:

\[
\bar{x}_i^{(n+1)} = (1 - \omega)\bar{x}_i^{(n)} + \omega A_{i,i}^{-1} \left[ \bar{r}_i - \sum_{j=1}^{i-1} A_{i,j} \bar{x}_j^{(n+1)} - \sum_{j=i+1}^{M} A_{i,j} \bar{x}_j^{(n)} \right],
\]

where \( A_{i,j} \) is the block corresponding to a single line of the numerical grid, \( M \) is the size of the matrix \( A \) in blocks, \( \bar{x}_i \) and \( \bar{r}_i \) are the subvectors of \( \bar{x} \) and \( \bar{r} \) respectively:

\[ A = \begin{pmatrix}
A_{1,1} & A_{1,2} & A_{1,M} \\
A_{2,1} & A_{2,2} & A_{2,3} & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
A_{M,1} & A_{M,M-1} & A_{M,M}
\end{pmatrix}
\]

\[
\bar{x} = \begin{pmatrix}
\bar{x}_1 \\
\bar{x}_2 \\
\vdots \\
\bar{x}_M
\end{pmatrix}; \quad \bar{r} = \begin{pmatrix}
\bar{r}_1 \\
\bar{r}_2 \\
\vdots \\
\bar{r}_M
\end{pmatrix}
\]

The formula 4.52 requires the inversion of the diagonal blocks \( A_{i,i} \). Fortunately, they are tridiagonal, and their inversion is not very time-consuming. Since the inversion is required at each iteration, \( LU \)-decomposition is applied to all diagonal blocks in the beginning of the iteration process, and the obtained matrices are stored in a form most suitable for the process 4.52.

Both point and block SOR methods are carried out using a relative error convergence criterion:

\[
\frac{||x^{(n+1)}|| - ||x^{(n)}||}{||x^{(n)}||} \leq \epsilon
\]

Usually \( \epsilon = 10^{-10} \div 10^{-7} \), and typically 10 \div 20 iterations are required by either of the two methods.
The problems considered in this work have the preferred direction of the shock wave propagation, and it can be used to improve the iteration process. Since the variation of all physical quantities in the $\xi$-direction is typically less than their variations in the preferred direction $\eta$, we use as preconditioner the matrix obtained from the 1D approximation of the energy equation 4.40. The diffusion operator 4.41 in 1D is

$$
\hat{D}T = \frac{\partial x}{\partial \xi} \frac{\partial}{\partial \eta} \frac{\kappa}{J} \frac{\partial x}{\partial \xi} \frac{\partial}{\partial \eta} T,
$$

and its difference approximation is

$$
(\hat{D}T)_{i+\frac{1}{2},j+\frac{1}{2}} = 
\frac{(x_\xi)_{i+\frac{1}{2},j+\frac{1}{2}}}{(\Delta \xi)(\Delta \eta)^2} \left[ \kappa_{i+\frac{1}{2},j+\frac{1}{2}} + \kappa_{i+\frac{1}{2},j-\frac{1}{2}} \right] (x_{i+1,j+1} - x_{i,j+1}) \left( T_{i+1,j+\frac{3}{2}} - T_{i+\frac{1}{2},j+\frac{3}{2}} \right) 
$$

$$
- \frac{\kappa_{i+\frac{1}{2},j+\frac{1}{2}}}{J_{i+\frac{1}{2},j+\frac{1}{2}}} + \frac{\kappa_{i+\frac{1}{2},j-\frac{1}{2}}}{J_{i+\frac{1}{2},j-\frac{1}{2}}} (x_{i+1,j} - x_{i,j}) \left( T_{i+\frac{1}{2},j+\frac{3}{2}} - T_{i+\frac{1}{2},j-\frac{3}{2}} \right),
$$

(4.57)

where $(x_\xi)_{i+\frac{1}{2},j+\frac{1}{2}}$ is defined by 4.27.

The iteration process

$$
\tilde{x}^{(n+1)} = \tilde{x}^{(n)} + \omega P^{-1} \left( \tilde{r} - A \tilde{x}^{(n)} \right)
$$

(4.58)

includes the matrix $P$ corresponding to the 1D approximation 4.57. $P$ has the block diagonal structure, and each block is tridiagonal. The inversion of such matrix consists of the successive or simultaneous use of the tridiagonal solver. Note that the 1D preconditioner is different from the one used in Jacobi iterations because the diagonal blocks $A_{i,i}$ of the original matrix $A$ do not exactly correspond to the one-dimensional approximation of the energy equation 4.40.

Using the preconditioner $P$ leads to a fast convergence of the iterations 4.58 as long as the problem is not strongly nonlinear. A combination of $P$ and Jacobi
Figure 4.11: The one-dimensional preconditioner matrix $P$ taking advantage of the preferred direction of the physical problem. Note that each diagonal block $P_{i,i}$ differs from the corresponding diagonal block $A_{i,i}$ in 4.53. The matrix $P$ is easy to invert using any standard tridiagonal solver.

The preconditioner $D$ gives

$$
\bar{x}^{(n+1)} = \bar{x}^{(n)} + \omega (\alpha P + (1-\alpha)D)^{-1} (\bar{r} - A\bar{x}^{(n)}),
$$

(4.59)

where $\alpha$ is adjustable and typically $0.3 < \alpha < 0.7$. The process 4.59 reduces the number of iteration by a factor of $2 \div 3$ relative to the SOR methods 4.51 and 4.52. Unfortunately, in some cases the iterations 4.58 and 4.59 diverge. Typically it occurs when the rippled shock or expansion wave is breaking out at the interface. A possible reason is the large condition number of the original matrix $A$. Since the matrix $P$ is built from physical intuition, the matrix $P^{-1}A$ is not proven to be positive definite, and in worst cases it is not.

The algorithm 4.59 can be used for the energy equation, and the program switches to either 4.51 or 4.52 when it fails.
4.1.7 Tests.

In order to test the numerical scheme, we have simulated the evolution of the classical RT instability at a gas/vacuum interface moving with constant acceleration, and the ablative RT instability at the front surface of a planar foil accelerated by an assigned heat source simulating the laser energy deposition at the critical surface. The classical RT instability simulation does not require the inclusion of heat conduction in the energy equation 4.7 and the boundary condition at the front surface is represented by the assigned pressure:

\[ P_{\text{ext}}(t, \xi) = A H(t), \quad (4.60) \]

where \( H(t) \) is the step function. The initial pressure is constant inside the foil and at its back surface. The front surface is initially perturbed, while the rear surface is flat. After the reflected expansion breaks out at the front surface, the acceleration of the surface relaxes to its asymptotic limit and becomes approximately constant (for \( t > 1\,\text{ns} \) in figure 4.12). A series of single mode simulations has been used to determine the classical RT growth rates and compared with the classical result \( \gamma = \sqrt{kg} \). The ablative RT instability is simulated by retaining the finite heat source in the energy equation 4.5 and its growth rate is compared with the result given in [10].

The qualitative behavior of the front surface is similar in both cases.

The numerical growth rates are obtained using the standard fitting procedure of the Mathematica software package (see [53]). The perturbation amplitude is fitted by the exponential law \( a = a_0 e^{\gamma t} \).

Figure 4.13 shows the growth rates obtained in both the classical and the ablative regimes. In both cases the front surface acceleration is \( g \approx 150\mu\text{m}/\text{ns}^2 \). The dots represent the numerical results, while the curves represent the theoretical
Figure 4.12: Typical front surface velocity and trajectory. The qualitative picture is the same in both classical and ablative regimes. Acceleration is quite constant after 1ns.

Figure 4.13: The growth rate tests. Solid curves are the theoretical predictions. Classical non-ablative growth rate is $\sqrt{kg}$, and the ablative growth rate is given in [10]. The square and round points are the numerical growth rates obtained in ablative and non-ablative regimes respectively.
growth rates. Figure 4.13 clearly indicates that the numerical results are in a good agreement with the theoretical predictions.

The other test is the evolution of the perturbation in a reflected expansion wave. An analytic theory of the process is developed in [40]. The problem solved by A. Velikovich and L. Phillips concerns the linear evolution of the perturbed hydro quantities within a rarefaction wave. The authors considered the rarefaction wave born in an interaction of the shock with the rippled interface between two fluids. The evolution of the perturbation at the leading edge of the rarefaction wave is characterized by an oscillatory spatial behavior. The pressure and velocity perturbations at the rarefaction leading edge obtained from the code are compared with the theory and shown in figure 4.14.

![Diagram of rarefaction wave and perturbations](image)

--- Theory --- Simulation

- At the leading edge perturbations oscillate in time.

Figure 4.14: Evolution of perturbation at the leading edge of rarefaction wave. $y$ is the direction of the wave propagation.

The numerical results shown in figure 4.14 represent the evolution of the perturbations in the hydrodynamic quantities defined as a fourier harmonics in the $x$—direction (parallel to the rarefaction front) near the leading edge.

4.2 Eulerian Code.

4.2.1 Motivation.

A second 2D code has been developed to simulate the deceleration phase of imploding capsules. Since the characteristic length scale does not change significantly during the deceleration phase, it is more appropriate to use an Eulerian description of fluid. The use of a rectangular Eulerian grid makes it easy to apply methods of splitting to the heat transfer equation and save computational time.

The code does not consider the laser light deposition (laser is assumed off during the deceleration phase). The $\alpha$-particle heating is included using the local deposition approximation without including particle transport. The energy losses due to bremsstrahlung radiation (but not a radiation transport) are included as a “sink” term in the energy equation.

The splitting by physical processes is performed in the code, e.g. the heat transfer solver is independent on the hydro equations.

4.2.2 Hydrodynamic Equations.

We use the conservative form of the fluid equations 2.2 - 2.4 can be transformed to the conservative form. In planar geometry the hydro equations become:

$$\frac{\partial}{\partial t} \tilde{U} + \frac{\partial}{\partial x} \tilde{F} + \frac{\partial}{\partial y} \tilde{G} = 0$$  \hspace{1cm} (4.61)
where $\vec{U}$, $\vec{F}$ and $\vec{G}$ are vectors given by

$$
\vec{U} = 
\begin{pmatrix}
\rho \\
\rho u_x \\
\rho u_y \\
\rho (E + \frac{u_x^2}{2})
\end{pmatrix}
$$

(4.62)

$$
\vec{F} = 
\begin{pmatrix}
\rho u_x \\
\rho u_x^2 + P \\
\rho u_x u_y \\
\left( \rho (E + \frac{u_x^2}{2}) + P \right) u_x
\end{pmatrix}
$$

(4.63)

$$
\vec{G} = 
\begin{pmatrix}
\rho u_y \\
\rho u_x u_y \\
\rho u_y^2 + P \\
\left( \rho (E + \frac{u_y^2}{2}) + P \right) u_y
\end{pmatrix}
$$

(4.64)

Note that, at this stage, neither the heat source nor the heat transfer terms are included in the equations. Such terms are taken into account in other parts of the program according to the splitting technique used here.

The left boundary at $x = 0$ represents a rigid wall, and the boundary conditions are

$$
\frac{\partial P}{\partial x} \bigg|_{x=0} = \frac{\partial \rho}{\partial x} \bigg|_{x=0} = \frac{\partial E}{\partial x} \bigg|_{x=0} = \frac{\partial u_y}{\partial x} \bigg|_{x=0} = 0, \quad u_x \bigg|_{x=0} = 0
$$

(4.65)

Since the flow is usually directed toward the left, we apply the zero-gradient boundary condition at the right boundary ($x = L_x$)

$$
\frac{\partial P}{\partial x} \bigg|_{x=L_x} = \frac{\partial \rho}{\partial x} \bigg|_{x=L_x} = \frac{\partial E}{\partial x} \bigg|_{x=L_x} = \frac{\partial u_y}{\partial x} \bigg|_{x=L_x} = \frac{\partial u_x}{\partial x} \bigg|_{x=L_x} = 0
$$

(4.66)

In the transverse direction the boundary conditions are either periodic

$$
f(x, 0) = f(x, L_y)
$$

(4.67)
or the Von Neumann boundary conditions:

\[
\frac{\partial f}{\partial y} \bigg|_{y=0} = \frac{\partial f}{\partial y} \bigg|_{y=L_y} = 0
\]  

(4.68)

where \(f(x, y)\) stands for any physical value.

In spherical geometry the 2D hydrodynamics equations with azimuthal symmetry are given by:

\[
\frac{\partial}{\partial t} \vec{U}_1 + \frac{\partial}{\partial r} \vec{F}_1 + \frac{\partial}{\partial \theta} \vec{G}_1 + \vec{H}_1 = 0
\]  

(4.69)

where the vectors \(\vec{U}_1, \vec{F}_1, \vec{G}_1\) and \(\vec{H}_1\) are

\[
\vec{U}_1 = \begin{pmatrix}
\rho \\
p \nu_r \\
p \nu_\theta \\
\rho (E + \frac{\nu_r^2}{2})
\end{pmatrix}
\]  

(4.70)

\[
\vec{F}_1 = \begin{pmatrix}
p \nu_r \\
p \nu_r^2 + P \\
p \nu_r \nu_\theta \\
\left(\rho (E + \frac{\nu_r^2}{2}) + P\right) \nu_r
\end{pmatrix}
\]  

(4.71)

\[
\vec{G}_1 = \frac{1}{r} \begin{pmatrix}
p \nu_\theta \\
p \nu_r \nu_\theta \\
p \nu_\theta^2 + P \\
\left(\rho (E + \frac{\nu_r^2}{2}) + P\right) \nu_\theta
\end{pmatrix}
\]  

(4.72)

\[
\vec{H}_1 = \frac{1}{r} \begin{pmatrix}
2p \nu_r + p \nu_\theta \cot(\theta) \\
2p \nu_r^2 + p \nu_r \nu_\theta \cot(\theta) - p \nu_\theta^2 \\
3p \nu_r \nu_\theta + p \nu_\theta^2 \cot(\theta) \\
\left(\rho (E + \frac{\nu_r^2}{2}) + P\right) \left(2 \nu_r + \nu_\theta \cot(\theta)\right)
\end{pmatrix}
\]  

(4.73)
4.2.3 Artificial Terms.

Though the MacCormack scheme (see, for example, [54] and Section 4.2.4 below) used in the codes is shock-capturing, it produces density oscillations at the shock front and contact discontinuities which need to be suppressed. In order to treat shock discontinuities, artificial viscosity terms similar to that described in section 4.1.3 are introduced as artificial pressures

\[
Q_x = a_0 \rho (\Delta v_x)^2 + a_L c_s \rho |\Delta v_z| \text{ if } \Delta v_x < 0, \\
Q_y = a_0 \rho (\Delta v_y)^2 + a_L c_s \rho |\Delta v_y| \text{ if } \Delta v_y < 0,
\]

where \(\Delta v_{x,y}\) is the change of the fluid velocity across the cell, \(a_0 \approx 4\) and \(a_L \approx 1\) are the adjustable constants, \(c_s\) and \(\rho\) are the local sound speed and density respectively. Unfortunately, the artificial viscosity terms do not suppress numerical oscillations near contact discontinuities because they vanish in the absence of velocity jumps. To suppress density oscillations at the shell edges, the following artificial heat flux is introduced:

\[
q_{x,y} = -\rho (b_1 |\Delta c_s| + b_2 c_s) \Delta E
\]

where \(\Delta c_s\) and \(\Delta E\) are the change of sound speed and internal energy across the cell in the specified direction, \(b_1\) and \(b_2\) are the adjustable constants (usually \(\leq 0.5\)).

The modified equations including the artificial terms are obtained by substituting 4.74 and 4.75 into the system 4.61 leading to the following modified expressions for \(\tilde{F}\) and \(\tilde{G}\):

\[
\tilde{F} = \begin{pmatrix}
\rho v_x \\
\rho v_x^2 + P + Q_x \\
\rho v_x v_y \\
\left(\rho (E + \frac{v_x^2}{2}) + P + Q_x\right) v_x + q_x
\end{pmatrix}
\]

(4.76)
Figure 4.15: Numerical grid in planar geometry

\[ \bar{G} = \begin{pmatrix} \rho v_y \\ \rho v_z v_y \\ \rho v_y^2 + P + Q_y \\ \left(\rho (E + \frac{\rho \theta^2}{2}) + P + Q_y\right) v_y + q_y \end{pmatrix} \] (4.77)

The same technique is applied in spherical geometry, \( Q_{r,\theta} \) and \( q_{r,\theta} \) being calculated similarly to 4.74, 4.75.

4.2.4 Numerical Solution of the Hydro Equations.

The planar equations 4.61 are being solved on a uniform rectangular grid (figure 4.15):

\[ x_i = i \Delta x, \quad i = 0, \ldots, N_x, \quad \Delta x = \frac{L_x}{N_x} \] (4.78)

\[ y_j = j \Delta y, \quad j = 0, \ldots, N_y, \quad \Delta y = \frac{L_y}{N_y} \] (4.79)

\[ f_{ij} = f(x_i, y_j) \] (4.80)

The finite difference scheme is based on the MacCormack algorithm shown
\[ Q_{xij} = a_0 \rho_{ij} (v_{xij} - v_{x_{i-1,j}})^2 + a_L c_{sij} \rho_{ij} |v_{xij} - v_{x_{i-1,j}}| \]  
if \( v_{xij} - v_{x_{i-1,j}} < 0, \)  
(4.81)

\[ Q_{yij} = a_0 \rho_{ij} (v_{yij} - v_{y_{i,j-1}})^2 + a_L c_{sij} \rho_{ij} |v_{yij} - v_{y_{i,j-1}}| \]  
if \( v_{yij} - v_{y_{i,j-1}} < 0, \)  
(4.82)

\[ q_{xij} = -\rho_{ij} (b_1 |c_{sij} - c_{s_{i-1,j}}| + b_2 c_{sij}) (E_{ij} - E_{i-1,j}) \]  
(4.83)

\[ q_{yij} = -\rho_{ij} (b_1 |c_{sij} - c_{s_{i,j-1}}| + b_2 c_{sij}) (E_{ij} - E_{i,j-1}) \]  
(4.84)

\[ \frac{\bar{U}_{i,j}^{n+1} - \bar{U}_{i,j}^n}{\Delta t} = -\frac{\bar{F}_{i+1,j}^n - \bar{F}_{i,j}^n}{\Delta x} - \frac{\bar{G}_{i+1,j}^n - \bar{G}_{i,j}^n}{\Delta y} \]  
(4.85)

\[ \bar{U}_{i,j}^{n+1} \Rightarrow \bar{F}_{i,j}^{n+1}, \bar{G}_{i,j}^{n+1} \]  
(4.86)

\[ \bar{Q}_{xij} = a_0 \bar{\rho}_{ij} (\bar{v}_{xi+1,j} - \bar{v}_{xij})^2 + a_L \bar{c}_{sij} \bar{\rho}_{ij} |\bar{v}_{xi+1,j} - \bar{v}_{xij}| \]  
if \( \bar{v}_{xi+1,j} - \bar{v}_{xij} < 0, \)  
(4.87)

\[ \bar{Q}_{yij} = a_0 \bar{\rho}_{ij} (\bar{v}_{y_{i+1,j}} - \bar{v}_{yij})^2 + a_L \bar{c}_{sij} \bar{\rho}_{ij} |\bar{v}_{y_{i+1,j}} - \bar{v}_{yij}| \]  
if \( \bar{v}_{y_{i+1,j}} - \bar{v}_{yij} < 0, \)  
(4.88)

\[ \bar{q}_{xij} = -\bar{\rho}_{ij} (b_1 |\bar{c}_{si+1,j} - \bar{c}_{sij}| + b_2 \bar{c}_{sij}) (\bar{E}_{i+1,j} - \bar{E}_{ij}) \]  
(4.89)

\[ \bar{q}_{yij} = -\bar{\rho}_{ij} (b_1 |\bar{c}_{s_{i+1,j}} - \bar{c}_{sij}| + b_2 \bar{c}_{sij}) (\bar{E}_{i,j+1} - \bar{E}_{ij}) \]  
(4.90)

\[ \frac{\bar{U}_{i,j}^{n+1} - \bar{U}_{i,j}^{n+1}}{\Delta t} = -\frac{\bar{F}_{i,j}^{n+1} - \bar{F}_{i-1,j}^{n+1}}{\Delta x} - \frac{\bar{G}_{i,j}^{n+1} - \bar{G}_{i,j-1}^{n+1}}{\Delta y} \]  
(4.91)

The superscript \( n \) is omitted in 4.81-4.84, as well as the superscript \( n + 1 \) in 4.87-4.90.
Treatment of the Von Neumann boundary conditions 4.65 - 4.66 and 4.68 is standard (see, for example, [55]) and includes the so called "ghost" points:

\[ f_{-1,j} = f_{1,j}, \quad f_{N_x+1,j} = f_{N_x-1,j}, \quad f_{i,-1} = f_{i,1}, \quad f_{i,N_y+1} = f_{i,N_y-1} \] (4.92)

where \( f \) stands for any physical value (except for \( v_x(0,y) \), of course). The "ghost" point values are substituted into the difference equations 4.81-4.91 at the boundaries. The Dirichlet condition \( v_x(0,y) = 0 \) is trivially satisfied. In the case of periodic boundary conditions 4.67 the obvious substitution is \( f_{i,-1} = f_{i,N_y}, \quad f_{i,N_y+1} = f_{i,0} \). The same boundary conditions are applied to both the predictor and corrector steps.

In spherical coordinates \((r, \theta)\) the grid is:

\[ r_i = i \Delta r, \quad i = 0, ..., N_r, \quad \Delta r = \frac{R}{N_r} \] (4.93)

\[ \theta_j = j \Delta \theta, \quad j = 0, ..., N_y, \quad \Delta \theta = \frac{\pi/2}{N_q} \] (4.94)

\[ f_{ij} = f \left( r_i, \theta_j \right) \] (4.95)

The domain 4.94 is the right angle sector (see figure 4.16). For many applications it is convenient to consider the smaller sector \( \theta_0 < \theta < \pi/2 \) with the zero normal derivatives along the sides \( \theta = \theta_0 \) and \( \theta = \pi/2 \).

The numerical algorithm similar to 4.81-4.91 is applied to the equation 4.69 in the internal points of the domain. The extrapolation is used at \( \theta = 0 \), if necessary. The center cell \( r = 0 \) is regarded to have zero velocity \( v_r = v_\theta = 0 \) due to the symmetry, and its density and energy can be found from the integral conservation laws:

\[ \frac{\partial}{\partial t} \int_{\text{cell}} \rho \, dV = - \int_{s} \rho v_r \, ds \] (4.96)

\[ \frac{\partial}{\partial t} \int_{\text{cell}} \rho \left( E + \frac{v^2}{2} \right) \, dV = - \int_{s} \left( \rho \left( E + \frac{v^2}{2} \right) + P \right) v_r \, ds \] (4.97)
where the right hand side integrals are over the center cell surface, \( ds \) is the surface area element, \( dV \) is the volume element.

Since \( dV = \left(4\pi/3\right)(\Delta r)^3 \), and \( ds = 2\pi (\Delta r)^2 \sin \theta \, d\theta \),

\[
\frac{\partial \rho}{\partial t} \bigg|_{r=0} = -\frac{3}{\Delta r} \int_0^{\pi/2} \rho v_r \sin \theta \, d\theta
\]

(4.98)

\[
\frac{\partial (\rho E)}{\partial t} \bigg|_{r=0} = -\frac{3}{\Delta r} \int_0^{\pi/2} \left( \rho \left( E + \frac{\overrightarrow{\mathbf{v}}^2}{2} \right) + P \right) v_r \sin \theta \, d\theta
\]

(4.99)

The treatment of the Von Neumann boundary conditions at \( \theta = \pi/2, \, \theta = \theta_0 \) (if \( \theta_0 \neq 0 \)), and \( r = R \) is similar to that used in the planar case (see 4.92 above).

### 4.2.5 Heat Transfer.

The heat diffusion equation can be written in the following simple form:

\[
\rho \frac{\partial}{\partial t} E = -\nabla \cdot \overrightarrow{\mathbf{q}}
\]

(4.100)

and describe the conduction energy transport in energy equation 2.4.

The heat flux \( \overrightarrow{\mathbf{q}} \) is represented according the Fourier conduction law:

\[
\overrightarrow{\mathbf{q}} = -\kappa \nabla E
\]

(4.101)
where

\[ \kappa = 3.8 \times 10^{-25} E^{5/2} \left[ \frac{kg}{m \cdot s} \left( \frac{J}{kg} \right)^{-5/2} \right] \]  

(4.102)

In planar geometry, the equation 4.100 becomes:

\[ \rho \frac{\partial}{\partial t} E = \frac{\partial}{\partial x} \kappa \frac{\partial}{\partial x} E + \frac{\partial}{\partial y} \kappa \frac{\partial}{\partial y} E \]  

(4.103)

with the boundary conditions

\[ \frac{\partial E}{\partial x} \bigg|_{x=0} = \frac{\partial E}{\partial x} \bigg|_{x=L_x} = \frac{\partial E}{\partial y} \bigg|_{y=0} = \frac{\partial E}{\partial y} \bigg|_{y=L_y} = 0 \]  

(4.104)

On the orthogonal grid 4.78 - 4.79, the equation 4.103 can be split to give two equations to be successively solved:

\[ \rho \frac{\partial}{\partial t} E = -\frac{\partial}{\partial x} \kappa \frac{\partial}{\partial x} E \]  

(4.105)

\[ \rho \frac{\partial}{\partial t} E = -\frac{\partial}{\partial y} \kappa \frac{\partial}{\partial y} E \]  

(4.106)

The implicit difference approximation of 4.105 is simply

\[ \tilde{E}_i \left[ \rho_i + \frac{\Delta t}{\Delta x^2} \frac{1}{2} (\kappa_{i-1} + 2\kappa_i + \kappa_{i+1}) \right] + \]  

\[ \tilde{E}_{i-1} \left[ -\frac{\Delta t}{\Delta x^2} \frac{1}{2} (\kappa_{i-1} + \kappa_i) \right] + \]  

\[ \tilde{E}_{i+1} \left[ -\frac{\Delta t}{\Delta x^2} \frac{1}{2} (\kappa_i + \kappa_{i+1}) \right] = \rho_i E_i^n \]  

(4.107)

where \( i = 1, \ldots, N_x - 1 \), and the second index \( j \) is omitted. At the boundary points \( i = 0 \) and \( i = N_x \), the difference equations are modified as follow to include the appropriate boundary conditions.

\( i = 0 \):

\[ \tilde{E}_i \left[ \rho_i + \frac{\Delta t}{\Delta x^2} (\kappa_i + \kappa_{i+1}) \right] + \tilde{E}_{i+1} \left[ -\frac{\Delta t}{\Delta x^2} (\kappa_i + \kappa_{i+1}) \right] = \rho_i E_i^n \]  

(4.108)

\( i = N_x \):

\[ \tilde{E}_i \left[ \rho_i + \frac{\Delta t}{\Delta x^2} (\kappa_i + \kappa_{i-1}) \right] + \tilde{E}_{i-1} \left[ -\frac{\Delta t}{\Delta x^2} (\kappa_{i-1} + \kappa_i) \right] = \rho_i E_i^n \]  

(4.109)
Similarly, the difference approximation of 4.106 can be written in the following form:

\[
    E^n_j \left[ \rho_j + \frac{\Delta t}{\Delta y^2} \frac{1}{2} (\bar{k}_{j-1} + 2\bar{k}_j + \bar{k}_{j+1}) \right] + \\
    E^n_{j-1} \left[ -\frac{\Delta t}{\Delta y^2} \frac{1}{2} (\bar{k}_{j-1} + \bar{k}_j) \right] + \\
    E^n_{j+1} \left[ -\frac{\Delta t}{\Delta y^2} \frac{1}{2} (\bar{k}_j + \bar{k}_{j+1}) \right] = \rho_j \bar{E}_j, \tag{4.110}
\]

where \( j = 1, \ldots, N_y - 1 \), the index \( i \) is omitted. At the boundary points \( j = 0 \) and \( j = N_y \), the boundary conditions require

\[ j = 0: \]

\[
    E^n_j \left[ \rho_j + \frac{\Delta t}{\Delta y^2} (\bar{k}_j + \bar{k}_{j+1}) \right] + E^n_{j+1} \left[ -\frac{\Delta t}{\Delta y^2} (\bar{k}_j + \bar{k}_{j+1}) \right] = \rho_j \bar{E}_j \tag{4.111}
\]

\[ j = N_y: \]

\[
    E^n_j \left[ \rho_j + \frac{\Delta t}{\Delta y^2} (\bar{k}_j + \bar{k}_{j-1}) \right] + E^n_{j-1} \left[ -\frac{\Delta t}{\Delta y^2} (\bar{k}_{j-1} + \bar{k}_j) \right] = \rho_j \bar{E}_j \tag{4.112}
\]

One should have noticed that the Von Neumann boundary conditions were used in the equations 4.108 - 4.109 for the intermediate value \( \bar{E} \).

The equations 4.107 - 4.109 and 4.110 - 4.112 define two linear algebraic systems with tridiagonal matrices, the bracketed expressions being the matrix coefficients. The tridiagonal matrices are solved by Thomas algorithm (see, for example, [55]). Using the direct solver is made possible due to the splitting by directions and demonstrates one of the advantages of the Eulerian algorithm over Lagrangian ones.

The equation 4.100 in spherical geometry is

\[
    \rho \frac{\partial E}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \kappa \frac{\partial E}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \kappa \sin \theta \frac{\partial E}{\partial \theta} \right) \tag{4.113}
\]
which also can be split into the radial and angular equations:

\[ r^2 \rho \frac{\partial}{\partial t} E = \frac{\partial}{\partial r} \left( r^2 \kappa \frac{\partial E}{\partial r} \right) \] (4.114)

\[ \rho \sin \theta \frac{\partial}{\partial t} E = \frac{1}{r} \frac{\partial}{\partial \theta} \left( \kappa \sin \theta \frac{1}{r} \frac{\partial E}{\partial \theta} \right) \] (4.115)

The radial difference equations are:

\[ \bar{E}_i \left[ r_i^2 \rho_i + \frac{\Delta t}{\Delta r^2} \frac{1}{2} (\chi_{i-1} + 2\chi_i + \chi_{i+1}) \right] + \]

\[ \bar{E}_{i-1} \left[ -\frac{\Delta t}{\Delta r^2} \frac{1}{2} (\chi_{i-1} + \chi_i) \right] + \]

\[ \bar{E}_{i+1} \left[ -\frac{\Delta t}{\Delta r^2} \frac{1}{2} (\chi_i + \chi_{i+1}) \right] = r_i^2 \rho_i E_i^n \] (4.116)

where \( \chi_i = r_i^2 \kappa_i, \ i = 1, ..., N_r - 1 \), the angular index \( j \) is omitted. To close up the linear system 4.116 add the two equations:

\[ i = 0 : \]

\[ \bar{E}_i = \bar{E}_{i+1} \] (4.117)

\[ i = N_r : \]

\[ \bar{E}_i \left[ r_i^2 \rho_i + \frac{\Delta t}{\Delta r^2} (\chi_i + \chi_{i-1}) \right] + \bar{E}_{i-1} \left[ -\frac{\Delta t}{\Delta r^2} (\chi_{i-1} + \chi_i) \right] = r_i^2 \rho_i E_i^n \] (4.118)

The difference equations in the poloidal angle \( \theta \) can be written as

\[ E_j^n \left[ \rho_j \sin \theta_j + \frac{\Delta t}{(r_j \Delta \theta)^2} \frac{1}{2} (\tilde{\eta}_{j-1} + 2\tilde{\eta}_j + \tilde{\eta}_{j+1}) \right] + \]

\[ E_{j-1}^n \left[ -\frac{\Delta t}{(r_j \Delta \theta)^2} \frac{1}{2} (\tilde{\eta}_{j-1} + \tilde{\eta}_j) \right] + \]

\[ E_{j+1}^n \left[ -\frac{\Delta t}{(r_j \Delta \theta)^2} \frac{1}{2} (\tilde{\eta}_j + \tilde{\eta}_{j+1}) \right] = \rho_j \sin \theta_j \tilde{E}_j, \] (4.119)

where \( \tilde{\eta}_j = \tilde{\kappa}_j \sin \theta_j, \ j = 1, ..., N_\theta - 1 \), the radial index \( i \) is omitted. The Neumann boundary conditions require that
\[ j = 0 : \quad E_j = E_{j+1} \]  

\[ j = N_q : \]

\[
E_j^{n} \left[ \rho_j + \frac{\Delta t}{(r_i \Delta \theta)^2} (\eta_j + \eta_{j-1}) \right] + E_{j-1}^{n} \left[ -\frac{\Delta t}{(r_i \Delta \theta)^2} (\eta_{j-1} + \eta_j) \right] = \rho_j \bar{E}_j \quad (4.121)
\]

This concludes the finite difference analysis of the conservation equations. The last step is to include the source terms in the splitting algorithm.

### 4.2.6 Source Terms.

The role of \( \alpha \)-particle deposition and energy losses due to bremsstrahlung radiation can be included in the energy equation 2.4 as source terms \( S \). The simplest model of the \( \alpha \)-particle deposition assumes that a fraction \( \Theta \) of all \( \alpha \)-particles born at some place is absorbed locally, without diffusion. Thus, the corresponding source term in the energy equation is simply the local alpha heating.

\[
S_\alpha = \Theta \frac{\rho^2 \epsilon_\alpha \langle \sigma v \rangle_{DT}}{4m_i^2} \quad (4.122)
\]

where \( \epsilon_\alpha \approx 3.5 MeV \) is the energy of a single \( \alpha \)-particle born in \( D + T \) fusion reaction, \( \rho \) is the local density, \( m_i \approx 2.5 m_p \) is the average mass of reacting elements, \( \langle \sigma v \rangle_{DT} \) is the reaction rate from the empirical table (see [30]). Since a goal of the simulation is to investigate hydrodynamic instabilities, rather than the ignition process, we use a reasonable expression for the absorbed fraction \( \theta \). Since energetic \( \alpha \)-particles leave the hot-spot when their mean-free path \( \lambda_\alpha \) is larger than the hot spot radius, it is reasonable to assume that the \( \alpha \)-particle fraction \( \Theta = R_{\text{hotspot}} / (R_{\text{hotspot}} + \lambda) \) is absorbed within the hot-spot, while the fraction \( (1 - \Theta) \) leaves the hot-spot.
Figure 4.17: The shell density profiles. (a) The filter 4.124 is used to suppress radiative instability in the shell. (b) No filter is used. The spike on the back surface is due to radiative instability.

The $\alpha$-particle mean-free path depends on the background plasma density and temperature. The form for $\lambda_\alpha$ used in the code is given by Lindl and reads as

$$\lambda_\alpha (gr/cm^2) = 1.5 \times 10^{-2} T^{5/4} / \left( \rho \left( 1 + 8.2 \cdot 10^{-3} T^{5/4} \right) \right),$$

where $T$ is measured in keV, $\rho$ is in $gr/cm^3$.

The bremsstrahlung radiation is taken into account as a simple sink term in the energy equation 2.4:

$$S_b = -9 \times 10^{10} \left[ \frac{m^3}{s \cdot kg} \left( \frac{J}{kg} \right)^{1/2} \right] \rho^2 E^{1/2},$$

(4.123)

implying that the radiation energy is entirely lost from the hot-spot at low temperature $\leq 4keV$, the bremsstrahlung radiation dominates over the $\alpha$-particle deposition, and an instability related to the radiation cooling may occur in the cold shell (see figure 4.17).
Such an instability does not occur in real ICF shell because of the finite optical depth, and it can be suppressed by a filter cutting the bremsstrahlung radiation at $E = E_{\text{cut}}$. In other words, we multiply the bremsstrahlung term in the energy equation by the factor $f$ shown below:

$$f = 1 - \exp \left\{ - \left( \frac{E}{E_{\text{cut}}} \right)^2 \right\} \quad (4.124)$$

In summary, the source terms contributing to the energy balance lead to the following last step of the splitting algorithm:

$$\rho \frac{\partial}{\partial t} E = S_\alpha + f S_b \quad (4.125)$$

### 4.2.7 Tests.

As a first test problem we have simulated a one-dimensional shock tube problem (see [56]). The initial conditions are shown in figure 4.18. We considered a 600$\mu$m domain with the discontinuity at $x = 300\mu$m. The two regions have different densities and pressures and are initially at rest. At $t = 0$ the shock front travels to the right, while the rarefaction wave propagates to the left.

The distributions of velocity, density and pressure given in figure 4.19 represent the shock wave propagating to the right, the rarefaction wave spreading to the left, and the contact surface where the density is discontinuous.

The other test problem is known as the Sedov Intense Explosion Problem (see [57]) which is appropriate to test the code's ability to model a strong radially diverging shock followed by an immediate rarefaction wave. The conservation of spherical symmetry is monitored throughout the simulation.

An ideal gas is initially uniform and at rest. An explosion takes place at the center of symmetry at zero time ($t = 0$). A strong spherical shock is expected to
Figure 4.18: The initial conditions in the shock tube problem.

propagate radially from the point of energy release. The evolution of the shock is considered by L.I.Sedov in [57], where the shock front trajectory as well as radial distributions of velocity, pressure and density behind the shock front are described analytically.

The simulated domain is 200\(\mu m\) in radius, the number of grid points in the radial direction is 500, the initial density is 50\(kg/m^3\), the initial pressure is 5 \(\cdot 10^5 Pa\) outside of the explosion. The initial pressure within the radius of 12\(\mu m\) is \(10^{12} Pa\). The fluid is initially at rest, i.e. the velocity components are equal to zero. The pressure, density and radial velocity profiles are compared with the analytic solution at \(t = 2.2 ns\).

Figure 4.20 represents the radial velocity, pressure and density distributions behind the shock front obtained both numerically and analytically.

The position of the shock front was monitored during the calculations. The Sedov solution and numerical evaluation of the shock position are represented in
Figure 4.19: The shock tube problem. The velocity, density and pressure distributions are given at $t = 2\mu s$. The dashed lines represent the exact analytic solution of the problem, while the dots represent the results of numerical simulation.
Figure 4.20: The intense explosion problem. The radial velocity, pressure and density distributions behind the shock front. The radius is normalized by the shock position, and the other values are in units of their peak values. Solid lines represent the numerical results while the dashed lines correspond to the analytic solution (see [57]).
Figure 4.21: The position of the shock front as a function of time. The dashed line is the Sedov solution, while the solid line is the computational result.

The explosion considered in [57] is 1D spherical, while the numerical tests are two-dimensional. Figure 4.22 shows the density contour plot at $t = 3.2\text{ns}$ after the explosion. The picture is drawn in the coordinates $(\theta, r)$ to demonstrate conservation of spherical symmetry. The absence of angular distortion is independent of the angular resolution.

4.3 Appendix. Matrix Coefficients

4.3.1 Heat Transfer Matrix Coefficients.

The energy equation 4.5 is approximated with the implicit difference scheme 4.42, in which the spatial difference operator is approximated on the new time step, and
Figure 4.22: Distribution of density in radial and angular directions at $t = 3.2\text{ns}$.

The backward difference is used for the time derivative. The spatial operator is approximated using the 9-points pattern.

Let us introduce the new functions $\Phi$ and $\Psi$:

\begin{equation}
\Phi = \frac{\kappa}{J} \left( \frac{\partial y}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial}{\partial \eta} \right) T
\end{equation}

\begin{equation}
\Psi = \frac{\kappa}{J} \left( \frac{\partial x}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial x}{\partial \xi} \frac{\partial}{\partial \eta} \right) T
\end{equation}

The differential operator 4.41 can be written as

\begin{equation}
\dot{\varphi} = \left\{ \left( \frac{\partial y}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial}{\partial \eta} \right) \frac{\kappa}{J} \left( \frac{\partial y}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial}{\partial \eta} \right) + \left( \frac{\partial x}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial x}{\partial \xi} \frac{\partial}{\partial \eta} \right) \frac{\kappa}{J} \left( \frac{\partial x}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial x}{\partial \xi} \frac{\partial}{\partial \eta} \right) \right\} T
\end{equation}

\begin{equation}
= \left( \frac{\partial y}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial}{\partial \eta} \right) \Phi + \left( \frac{\partial x}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial x}{\partial \xi} \frac{\partial}{\partial \eta} \right) \Psi,
\end{equation}

(4.128)
Figure 4.23: The points used in the approximation of the energy equation.

and its difference approximation is

\[
(\hat{DT})_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2\Delta \xi} \left( y_\eta \right)_{i+\frac{1}{2},j+\frac{1}{2}} \left( \Phi_{i+1,j+1} - \Phi_{i,j+1} + \Phi_{i+1,j} - \Phi_{i,j} \right) \\
- \frac{1}{2\Delta \eta} \left( y_\xi \right)_{i+\frac{1}{2},j+\frac{1}{2}} \left( \Phi_{i+1,j+1} - \Phi_{i+1,j} + \Phi_{i,j+1} - \Phi_{i,j} \right) \\
+ \frac{1}{2\Delta \eta} \left( x_\xi \right)_{i+\frac{1}{2},j+\frac{1}{2}} \left( \Psi_{i+1,j+1} - \Psi_{i+1,j} + \Psi_{i,j+1} - \Psi_{i,j} \right) \\
- \frac{1}{2\Delta \xi} \left( x_\eta \right)_{i+\frac{1}{2},j+\frac{1}{2}} \left( \Psi_{i+1,j+1} - \Psi_{i+1,j} + \Psi_{i,j+1} - \Psi_{i,j} \right) \tag{4.129}
\]

\((x_\xi)_{i+\frac{1}{2},j+\frac{1}{2}}, (x_\eta)_{i+\frac{1}{2},j+\frac{1}{2}}, (y_\xi)_{i+\frac{1}{2},j+\frac{1}{2}}\) and \((y_\eta)_{i+\frac{1}{2},j+\frac{1}{2}}\) are defined by 4.27-4.30. \(\Phi_{ij}\) and \(\Psi_{ij}\) are the difference approximations of the functions \(\Phi\) and \(\Psi\) at the point \((i,j)\).
The difference approximation 4.129 requires knowing the values of functions $\Phi$ and $\Psi$ at the points $(i, j), (i + 1, j), (i, j + 1)$ and $(i + 1, j + 1)$:

\[
\Phi_{i,j} = \left(\frac{\kappa}{J}\right)_{i,j} \frac{1}{4 \Delta \xi \Delta \eta} \times \\
\left\{ (y_{i,j+1} - y_{i,j-1}) \left( T_{i+\frac{1}{2},j+\frac{1}{2}} - T_{i-\frac{1}{2},j+\frac{1}{2}} + T_{i+\frac{1}{2},j-\frac{1}{2}} - T_{i-\frac{1}{2},j-\frac{1}{2}} \right) - \\
(y_{i+1,j} - y_{i-1,j}) \left( T_{i+\frac{1}{2},j+\frac{1}{2}} - T_{i+\frac{1}{2},j-\frac{1}{2}} + T_{i-\frac{1}{2},j+\frac{1}{2}} - T_{i-\frac{1}{2},j-\frac{1}{2}} \right) \right\} \quad (4.130)
\]

\[
\Phi_{i+1,j} = \left(\frac{\kappa}{J}\right)_{i+1,j} \frac{1}{4 \Delta \xi \Delta \eta} \times \\
\left\{ (y_{i+1,j+1} - y_{i+1,j-1}) \left( T_{i+\frac{3}{2},j+\frac{1}{2}} - T_{i+\frac{1}{2},j+\frac{1}{2}} + T_{i+\frac{5}{2},j-\frac{1}{2}} - T_{i+\frac{3}{2},j-\frac{1}{2}} \right) - \\
(y_{i+2,j} - y_{i,j}) \left( T_{i+\frac{1}{2},j+\frac{1}{2}} - T_{i+\frac{3}{2},j-\frac{1}{2}} + T_{i+\frac{1}{2},j+\frac{1}{2}} - T_{i+\frac{3}{2},j-\frac{1}{2}} \right) \right\} \quad (4.131)
\]

\[
\Phi_{i,j+1} = \left(\frac{\kappa}{J}\right)_{i,j+1} \frac{1}{4 \Delta \xi \Delta \eta} \times \\
\left\{ (y_{i,j+2} - y_{i,j}) \left( T_{i+\frac{1}{2},j+\frac{3}{2}} - T_{i-\frac{1}{2},j+\frac{3}{2}} + T_{i+\frac{1}{2},j+\frac{1}{2}} - T_{i-\frac{1}{2},j+\frac{1}{2}} \right) - \\
(y_{i+1,j+1} - y_{i-1,j+1}) \left( T_{i+\frac{3}{2},j+\frac{1}{2}} - T_{i+\frac{3}{2},j+\frac{1}{2}} + T_{i-\frac{1}{2},j+\frac{3}{2}} - T_{i-\frac{1}{2},j+\frac{3}{2}} \right) \right\} \quad (4.132)
\]

\[
\Phi_{i+1,j+1} = \left(\frac{\kappa}{J}\right)_{i+1,j+1} \frac{1}{4 \Delta \xi \Delta \eta} \times \\
\left\{ (y_{i+1,j+2} - y_{i+1,j}) \left( T_{i+\frac{3}{2},j+\frac{3}{2}} - T_{i+\frac{1}{2},j+\frac{3}{2}} + T_{i+\frac{3}{2},j+\frac{1}{2}} - T_{i+\frac{1}{2},j+\frac{1}{2}} \right) - \\
(y_{i+2,j+1} - y_{i,j+1}) \left( T_{i+\frac{3}{2},j+\frac{1}{2}} - T_{i+\frac{1}{2},j+\frac{3}{2}} + T_{i+\frac{1}{2},j+\frac{3}{2}} - T_{i+\frac{1}{2},j+\frac{3}{2}} \right) \right\} \quad (4.133)
\]

\[
\Psi_{i,j} = \left(\frac{\kappa}{J}\right)_{i,j} \frac{1}{4 \Delta \xi \Delta \eta} \times \\
\left\{ (x_{i+1,j} - x_{i-1,j}) \left( T_{i+\frac{1}{2},j+\frac{1}{2}} - T_{i+\frac{1}{2},j-\frac{1}{2}} + T_{i-\frac{1}{2},j+\frac{1}{2}} - T_{i-\frac{1}{2},j-\frac{1}{2}} \right) - \\
(x_{i,j+1} - x_{i,j-1}) \left( T_{i+\frac{1}{2},j+\frac{1}{2}} - T_{i-\frac{1}{2},j+\frac{1}{2}} + T_{i+\frac{1}{2},j-\frac{1}{2}} - T_{i-\frac{1}{2},j-\frac{1}{2}} \right) \right\} \quad (4.134)
\]
\[ \Psi_{i+1,j} = \left( \frac{\kappa}{J} \right)_{i+1,j} \frac{1}{4 \Delta \xi \Delta \eta} \times \]
\[ \left\{ (x_{i+2,j} - x_{i,j}) \left( T_{i+\frac{3}{2}, j+\frac{1}{2}} - T_{i+\frac{3}{2}, j-\frac{1}{2}} + T_{i+\frac{1}{2}, j+\frac{1}{2}} - T_{i+\frac{1}{2}, j-\frac{1}{2}} \right) - \\
(x_{i+1,j+1} - y_{i+1,j-1}) \left( T_{i+\frac{3}{2}, j+\frac{1}{2}} - T_{i+\frac{3}{2}, j-\frac{1}{2}} + T_{i+\frac{1}{2}, j+\frac{1}{2}} - T_{i+\frac{1}{2}, j-\frac{1}{2}} \right) \right\} \] (4.135)

\[ \Psi_{i,j+1} = \left( \frac{\kappa}{J} \right)_{i,j+1} \frac{1}{4 \Delta \xi \Delta \eta} \times \]
\[ \left\{ (x_{i+1,j+1} - x_{i-1,j+1}) \left( T_{i+\frac{3}{2}, j+\frac{1}{2}} - T_{i+\frac{1}{2}, j+\frac{1}{2}} + T_{i-\frac{1}{2}, j+\frac{1}{2}} - T_{i-\frac{1}{2}, j+\frac{1}{2}} \right) - \\
(x_{i,j+2} - y_{i,j}) \left( T_{i+\frac{1}{2}, j+\frac{3}{2}} - T_{i-\frac{1}{2}, j+\frac{3}{2}} + T_{i+\frac{1}{2}, j+\frac{1}{2}} - T_{i-\frac{1}{2}, j+\frac{1}{2}} \right) \right\} \] (4.136)

\[ \Psi_{i+1,j+1} = \left( \frac{\kappa}{J} \right)_{i+1,j+1} \frac{1}{4 \Delta \xi \Delta \eta} \times \]
\[ \left\{ (x_{i+2,j+1} - x_{i,j+1}) \left( T_{i+\frac{3}{2}, j+\frac{1}{2}} - T_{i+\frac{3}{2}, j+\frac{1}{2}} + T_{i+\frac{1}{2}, j+\frac{1}{2}} - T_{i+\frac{1}{2}, j+\frac{1}{2}} \right) - \\
(x_{i+1,j+2} - y_{i+1,j}) \left( T_{i+\frac{3}{2}, j+\frac{3}{2}} - T_{i+\frac{1}{2}, j+\frac{3}{2}} + T_{i+\frac{3}{2}, j+\frac{1}{2}} - T_{i+\frac{1}{2}, j+\frac{1}{2}} \right) \right\} \] (4.137)

The term \((\kappa/J)\) in equations 4.130-4.137 is

\[ \left( \frac{\kappa}{J} \right)_{i,j} = \frac{\kappa_{i-\frac{1}{2}, j-\frac{1}{2}} + \kappa_{i-\frac{1}{2}, j+\frac{1}{2}} + \kappa_{i+\frac{1}{2}, j-\frac{1}{2}} + \kappa_{i+\frac{1}{2}, j+\frac{1}{2}}}{J_{i-\frac{1}{2}, j-\frac{1}{2}} + J_{i-\frac{1}{2}, j+\frac{1}{2}} + J_{i+\frac{1}{2}, j-\frac{1}{2}} + J_{i+\frac{1}{2}, j+\frac{1}{2}}} \] (4.138)

Substitution of 4.129-4.137 into 4.42 gives the linear set of equations to be solved:

\[ \sum_{k=i-1}^{i+1} \sum_{l=j-1}^{j+1} a_{kl}^{ij} T_{k+\frac{1}{2}, l+\frac{1}{2}}^{n+1} = r_{ij}, \] (4.139)

\[ i = 0, \ldots, N_{\xi} - 1, j = 0, \ldots, N_{\eta} - 1 \]

The meaning of the indexes in \(a_{kl}^{ij}\) can be understood from the figure 4.24: \((i + 1)\) is the vertical number of the block, and \((j + 1)\) is the line number in the block.
Figure 4.24: Structure of the matrix A.

\((k + 1)\) is the horizontal number of the block, \((l + 1)\) is the column number in the block.

The right hand side of the equation 4.139 is the vector \(\vec{r}\) with the components:

\[
\begin{align*}
r_{ij} &= \frac{c_u}{\Delta t} (\rho_0 J_0)_{i+\frac{1}{2},j+\frac{1}{2}} \left(1 - (\Gamma - 1) \frac{J^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} - J^{n}_{i+\frac{1}{2},j+\frac{1}{2}}}{J^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} + J^{n}_{i+\frac{1}{2},j+\frac{1}{2}}} \right) T^n_{i+\frac{1}{2},j+\frac{1}{2}} \\
&+ \frac{S^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} J^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} + S^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} J^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}}{2} \\
&- \frac{Q^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} J^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} - J^{n}_{i+\frac{1}{2},j+\frac{1}{2}}}{\Delta t}
\end{align*}
\]

(4.140)

The matrix coefficients are obtained by grouping the corresponding terms in
4.129-4.137 and 4.42.

Let us define

\[(\delta_i x)_{i,j} = x_{i+1,j} - x_{i-1,j}\]  
(4.141)

\[(\delta_j x)_{i,j} = x_{i,j+1} - x_{i,j-1}\]  
(4.142)

\[(\delta_i y)_{i,j} = y_{i+1,j} - y_{i-1,j}\]  
(4.143)

\[(\delta_j y)_{i,j} = y_{i,j+1} - y_{i,j-1}\]  
(4.144)

One can also put \(\Delta \xi = 1, \Delta \eta = 1\) without loss of generality.

Now let us write down the matrix elements. From now on the super-indexes in \(a_{ij}^{\eta}\) are dropped and assumed \(i, j\) unless stated otherwise. Thus the coefficients represented here correspond to a single line in the matrix, namely the line number \((i \cdot N_n + j + 1)\). It can also be noted, that this is the \((j + 1)\)-th line of the \((i + 1)\)-th block of the matrix \(A\) in 4.43.

Consider first the internal points of the domain: \(i = 1, ..., N_\xi - 2, j = 1, ..., N_\eta - 2\). The diagonal element \(a_{ij}\) is

\[a_{ij} = \frac{c \nu}{\Delta t} (\rho_0 J_0)_{i+\frac{1}{2}, j+\frac{1}{2}} \left( 1 + (\Gamma - 1) \frac{J_{i+1,j+\frac{1}{2}}}{J_{i+\frac{1}{2}, j+\frac{1}{2}}} - \frac{J_{i+\frac{1}{2}, j+\frac{1}{2}}}{J_{i+\frac{1}{2}, j+\frac{1}{2}}} \right) - \frac{1}{8} \left( \frac{\kappa}{\nu} \right)_{i,j} \left\{ \left[ (y_\xi)_{i+\frac{1}{2}, j+\frac{1}{2}} - (y_\eta)_{i+\frac{1}{2}, j+\frac{1}{2}} \right] \left[ (\delta_j y)_{i,j} - (\delta_i y)_{i,j} \right] + \left[ (x_\eta)_{i+\frac{1}{2}, j+\frac{1}{2}} - (x_\xi)_{i+\frac{1}{2}, j+\frac{1}{2}} \right] \left[ (\delta_i x)_{i,j} - (\delta_j x)_{i,j} \right] \right\}
- \frac{1}{8} \left( \frac{\kappa}{\nu} \right)_{i+1,j} \left\{ \left[ (y_\xi)_{i+\frac{1}{2}, j+\frac{1}{2}} + (y_\eta)_{i+\frac{1}{2}, j+\frac{1}{2}} \right] \left[ - (\delta_j y)_{i+1,j} - (\delta_i y)_{i+1,j} \right] + \left[ (x_\eta)_{i+\frac{1}{2}, j+\frac{1}{2}} + (x_\xi)_{i+\frac{1}{2}, j+\frac{1}{2}} \right] \left[ -(\delta_i x)_{i+1,j} - (\delta_j x)_{i+1,j} \right] \right\}
- \frac{1}{8} \left( \frac{\kappa}{\nu} \right)_{i,j+1} \left\{ \left[ (y_\xi)_{i+\frac{1}{2}, j+\frac{1}{2}} + (y_\eta)_{i+\frac{1}{2}, j+\frac{1}{2}} \right] \left[ - (\delta_j y)_{i,j+1} - (\delta_i y)_{i,j+1} \right] + \left[ (x_\eta)_{i+\frac{1}{2}, j+\frac{1}{2}} + (x_\xi)_{i+\frac{1}{2}, j+\frac{1}{2}} \right] \left[ -(\delta_i x)_{i,j+1} - (\delta_j x)_{i,j+1} \right] \right\}
- \frac{1}{8} \left( \frac{\kappa}{\nu} \right)_{i+1,j+1} \left\{ \left[ (y_\xi)_{i+\frac{1}{2}, j+\frac{1}{2}} - (y_\eta)_{i+\frac{1}{2}, j+\frac{1}{2}} \right] \left[ (\delta_j y)_{i+1,j+1} - (\delta_i y)_{i+1,j+1} \right] + \left[ (x_\eta)_{i+\frac{1}{2}, j+\frac{1}{2}} - (x_\xi)_{i+\frac{1}{2}, j+\frac{1}{2}} \right] \left[ (\delta_i x)_{i+1,j+1} - (\delta_j x)_{i+1,j+1} \right] \right\} \quad (4.145)\]
The off-diagonal elements are:

\[
a_{i-1,j} = \frac{1}{8} \left( \frac{\kappa}{J} \right)_{i,j} \left\{ \left[ (y_\xi)_{i+\frac{1}{2}J+\frac{1}{2}} - (y_\eta)_{i+\frac{1}{2}J+\frac{1}{2}} \right] \left[ (\delta_\xi y)_{i,j} + (\delta_\eta y)_{i,j} \right] + \left[ (x_\xi)_{i+\frac{1}{2}J+\frac{1}{2}} - (x_\eta)_{i+\frac{1}{2}J+\frac{1}{2}} \right] \left[ (\delta_\xi x)_{i,j} + (\delta_\eta x)_{i,j} \right] - \frac{1}{8} \left( \frac{\kappa}{J} \right)_{i,j+1} \left\{ \left[ (y_\xi)_{i+\frac{1}{2}J+\frac{1}{2}} + (y_\eta)_{i+\frac{1}{2}J+\frac{1}{2}} \right] \left[ (\delta_\xi y)_{i,j+1} - (\delta_\eta y)_{i,j+1} \right] + \left[ (x_\xi)_{i+\frac{1}{2}J+\frac{1}{2}} + (x_\eta)_{i+\frac{1}{2}J+\frac{1}{2}} \right] \left[ (\delta_\xi x)_{i,j+1} - (\delta_\eta x)_{i,j+1} \right] \right\} \right) \right. \tag{4.146}
\]

\[
a_{i,j-1} = \frac{1}{8} \left( \frac{\kappa}{J} \right)_{i,j} \left\{ \left[ (y_\eta)_{i+\frac{1}{2}J+\frac{1}{2}} - (y_\xi)_{i+\frac{1}{2}J+\frac{1}{2}} \right] \left[ (\delta_\eta y)_{i,j} + (\delta_\xi y)_{i,j} \right] - \left[ (x_\eta)_{i+\frac{1}{2}J+\frac{1}{2}} - (x_\xi)_{i+\frac{1}{2}J+\frac{1}{2}} \right] \left[ (\delta_\eta x)_{i,j} + (\delta_\xi x)_{i,j} \right] + \frac{1}{8} \left( \frac{\kappa}{J} \right)_{i+1,j} \left\{ \left[ (y_\xi)_{i+\frac{1}{2}J+\frac{1}{2}} + (y_\eta)_{i+\frac{1}{2}J+\frac{1}{2}} \right] \left[ (\delta_\xi y)_{i+1,j} - (\delta_\eta y)_{i+1,j} \right] + \left[ (x_\xi)_{i+\frac{1}{2}J+\frac{1}{2}} + (x_\eta)_{i+\frac{1}{2}J+\frac{1}{2}} \right] \left[ (\delta_\xi x)_{i+1,j} - (\delta_\eta x)_{i+1,j} \right] \right\} \right) \right. \tag{4.147}
\]

\[
a_{i,j+1} = \frac{1}{8} \left( \frac{\kappa}{J} \right)_{i,j+1} \left\{ \left[ (y_\xi)_{i+\frac{1}{2}J+\frac{1}{2}} + (y_\eta)_{i+\frac{1}{2}J+\frac{1}{2}} \right] \left[ (\delta_\xi y)_{i,j+1} - (\delta_\eta y)_{i,j+1} \right] + \left[ (x_\xi)_{i+\frac{1}{2}J+\frac{1}{2}} + (x_\eta)_{i+\frac{1}{2}J+\frac{1}{2}} \right] \left[ (\delta_\xi x)_{i,j+1} - (\delta_\eta x)_{i,j+1} \right] + \frac{1}{8} \left( \frac{\kappa}{J} \right)_{i+1,j+1} \left\{ \left[ (y_\eta)_{i+\frac{1}{2}J+\frac{1}{2}} - (y_\xi)_{i+\frac{1}{2}J+\frac{1}{2}} \right] \left[ (\delta_\xi y)_{i+1,j+1} + (\delta_\eta y)_{i+1,j+1} \right] + \left[ (x_\eta)_{i+\frac{1}{2}J+\frac{1}{2}} - (x_\xi)_{i+\frac{1}{2}J+\frac{1}{2}} \right] \left[ (\delta_\xi x)_{i+1,j+1} + (\delta_\eta x)_{i+1,j+1} \right] \right\} \right) \right. \tag{4.148}
\]

\[
a_{i+1,j} = \frac{1}{8} \left( \frac{\kappa}{J} \right)_{i+1,j} \left\{ \left[ (y_\eta)_{i+\frac{1}{2}J+\frac{1}{2}} + (y_\xi)_{i+\frac{1}{2}J+\frac{1}{2}} \right] \left[ (\delta_\xi y)_{i+1,j} - (\delta_\eta y)_{i+1,j} \right] + \left[ (x_\eta)_{i+\frac{1}{2}J+\frac{1}{2}} + (x_\xi)_{i+\frac{1}{2}J+\frac{1}{2}} \right] \left[ (\delta_\xi x)_{i+1,j} - (\delta_\eta x)_{i+1,j} \right] - \frac{1}{8} \left( \frac{\kappa}{J} \right)_{i+1,j+1} \left\{ \left[ (y_\xi)_{i+\frac{1}{2}J+\frac{1}{2}} - (y_\eta)_{i+\frac{1}{2}J+\frac{1}{2}} \right] \left[ (\delta_\xi y)_{i+1,j+1} + (\delta_\eta y)_{i+1,j+1} \right] + \left[ (x_\xi)_{i+\frac{1}{2}J+\frac{1}{2}} - (x_\eta)_{i+\frac{1}{2}J+\frac{1}{2}} \right] \left[ (\delta_\xi x)_{i+1,j+1} + (\delta_\eta x)_{i+1,j+1} \right] \right\} \right) \right. \tag{4.149}
\]
\[ a_{i-1,j-1} = \frac{1}{8} \left( \frac{\kappa}{J} \right)_{i,j} \left\{ \left[ (y_{\xi})_{i+\frac{1}{2},j+\frac{1}{2}} - (y_{\eta})_{i+\frac{1}{2},j+\frac{1}{2}} \right] \left[ (\delta_j y)_{i,j} - (\delta_i y)_{i,j} \right] \\
+ \left[ (x_{\xi})_{i+\frac{1}{2},j+\frac{1}{2}} - (x_{\eta})_{i+\frac{1}{2},j+\frac{1}{2}} \right] \left[ (\delta_j x)_{i,j} + (\delta_i x)_{i,j} \right] \right\} \] (4.150)

\[ a_{i-1,j+1} = -\frac{1}{8} \left( \frac{\kappa}{J} \right)_{i,j+1} \left\{ \left[ (y_{\xi})_{i+\frac{1}{2},j+\frac{1}{2}} + (y_{\eta})_{i+\frac{1}{2},j+\frac{1}{2}} \right] \left[ (\delta_j y)_{i,j+1} + (\delta_i y)_{i,j+1} \right] \\
+ \left[ (x_{\xi})_{i+\frac{1}{2},j+\frac{1}{2}} + (x_{\eta})_{i+\frac{1}{2},j+\frac{1}{2}} \right] \left[ (\delta_j x)_{i,j+1} + (\delta_i x)_{i,j+1} \right] \right\} \] (4.151)

\[ a_{i+1,j-1} = -\frac{1}{8} \left( \frac{\kappa}{J} \right)_{i+1,j} \left\{ \left[ (y_{\xi})_{i+\frac{1}{2},j+\frac{1}{2}} + (y_{\eta})_{i+\frac{1}{2},j+\frac{1}{2}} \right] \left[ (\delta_j y)_{i+1,j} + (\delta_i y)_{i+1,j} \right] \\
+ \left[ (x_{\xi})_{i+\frac{1}{2},j+\frac{1}{2}} + (x_{\eta})_{i+\frac{1}{2},j+\frac{1}{2}} \right] \left[ (\delta_j x)_{i+1,j} + (\delta_i x)_{i+1,j} \right] \right\} \] (4.152)

\[ a_{i+1,j+1} = \frac{1}{8} \left( \frac{\kappa}{J} \right)_{i+1,j+1} \left\{ \left[ (y_{\xi})_{i+\frac{1}{2},j+\frac{1}{2}} - (y_{\eta})_{i+\frac{1}{2},j+\frac{1}{2}} \right] \left[ (\delta_j y)_{i+1,j+1} - (\delta_i y)_{i+1,j+1} \right] \\
+ \left[ (x_{\xi})_{i+\frac{1}{2},j+\frac{1}{2}} - (x_{\eta})_{i+\frac{1}{2},j+\frac{1}{2}} \right] \left[ (\delta_j x)_{i+1,j+1} + (\delta_i x)_{i+1,j+1} \right] \right\} \] (4.153)

The boundary points \( i = 0 \) are treated according to the periodic boundary conditions \( T_{-1/2,j} = T_{N_x-1/2,j} \). Thus, the elements \( a_{0,j}^{0,j}, a_{0,j+1}^{0,j}, a_{0,j}^{0,j+1}, a_{0,j+1}^{0,j} \) are undefined, but instead

\[ a_{N_x-1,j}^{0,j} = \frac{1}{8} \left( \frac{\kappa}{J} \right)_{0,j} \left\{ \left[ (y_{\xi})_{\frac{1}{2},j+\frac{1}{2}} - (y_{\eta})_{\frac{1}{2},j+\frac{1}{2}} \right] \left[ (\delta_j y)_{0,j} + (\delta_i y)_{0,j} \right] \\
+ \left[ (x_{\xi})_{\frac{1}{2},j+\frac{1}{2}} - (x_{\eta})_{\frac{1}{2},j+\frac{1}{2}} \right] \left[ (\delta_j x)_{0,j} + (\delta_i x)_{0,j} \right] \right\} \\
- \frac{1}{8} \left( \frac{\kappa}{J} \right)_{0,j+1} \left\{ \left[ (y_{\xi})_{\frac{1}{2},j+\frac{1}{2}} + (y_{\eta})_{\frac{1}{2},j+\frac{1}{2}} \right] \left[ (\delta_j y)_{0,j+1} + (\delta_i y)_{0,j+1} \right] \\
+ \left[ (x_{\xi})_{\frac{1}{2},j+\frac{1}{2}} + (x_{\eta})_{\frac{1}{2},j+\frac{1}{2}} \right] \left[ (\delta_j x)_{0,j+1} + (\delta_i x)_{0,j+1} \right] \right\} \] (4.154)

\[ a_{N_x-1,j-1}^{0,j} = \frac{1}{8} \left( \frac{\kappa}{J} \right)_{0,j} \left\{ \left[ (y_{\xi})_{\frac{1}{2},j+\frac{1}{2}} - (y_{\eta})_{\frac{1}{2},j+\frac{1}{2}} \right] \left[ (\delta_j y)_{0,j} - (\delta_i y)_{0,j} \right] \\
+ \left[ (x_{\xi})_{\frac{1}{2},j+\frac{1}{2}} - (x_{\eta})_{\frac{1}{2},j+\frac{1}{2}} \right] \left[ (\delta_j x)_{0,j} + (\delta_i x)_{0,j} \right] \right\} \\
+ \frac{1}{8} \left( \frac{\kappa}{J} \right)_{0,j+1} \left\{ \left[ (y_{\xi})_{\frac{1}{2},j+\frac{1}{2}} + (y_{\eta})_{\frac{1}{2},j+\frac{1}{2}} \right] \left[ (\delta_j y)_{0,j+1} - (\delta_i y)_{0,j+1} \right] \\
+ \left[ (x_{\xi})_{\frac{1}{2},j+\frac{1}{2}} - (x_{\eta})_{\frac{1}{2},j+\frac{1}{2}} \right] \left[ (\delta_j x)_{0,j+1} + (\delta_i x)_{0,j+1} \right] \right\} \] (4.155)
\[ a_{N_{\xi}, 0, j+1}^{0,j} = -\frac{1}{8} \left( \frac{\kappa}{f} \right)_{0,j+1} \left\{ \left[ (y_{\xi})_{\frac{1}{2}, j+\frac{1}{2}} + (y_{\eta})_{\frac{1}{2}, j+\frac{1}{2}} \right] \left[ (\delta_i y)_{0,j+1} + (\delta_i y)_{0,j+1} \right] \\
+ \left[ (x_{\xi})_{\frac{1}{2}, j+\frac{1}{2}} + (x_{\eta})_{\frac{1}{2}, j+\frac{1}{2}} \right] \left[ (\delta_j x)_{0,j+1} + (\delta_j x)_{0,j+1} \right] \right\} \quad (4.156) \]

The right boundary points \( i = N_{\xi} \) are treated in the same way. The periodic boundary condition is \( T_{1/2,j} = T_{N_{\xi}+1/2,j} \). The elements \( a_{N_{\xi}, 0,j}^{N_{\xi}-1,j}, a_{N_{\xi}, 0,j-1}^{N_{\xi}-1,j}, a_{N_{\xi}, 0,j+1}^{N_{\xi}-1,j} \) are undefined, and

\[ a_{0,j}^{N_{\xi}-1,j} = \frac{1}{8} \left( \frac{\kappa}{f} \right)_{0,j} \left\{ \left[ (y_{\xi})_{N_{\xi} - \frac{1}{2}, j+\frac{1}{2}} + (y_{\eta})_{N_{\xi} - \frac{1}{2}, j+\frac{1}{2}} \right] \left[ (\delta_i y)_{0,j} - (\delta_j y)_{0,j} \right] \\
+ \left[ (x_{\xi})_{N_{\xi} - \frac{1}{2}, j+\frac{1}{2}} + (x_{\eta})_{N_{\xi} - \frac{1}{2}, j+\frac{1}{2}} \right] \left[ (\delta_i x)_{0,j} - (\delta_j x)_{0,j} \right] \right\} \quad (4.157) \]

\[ a_{0,j-1}^{N_{\xi}-1,j} = -\frac{1}{8} \left( \frac{\kappa}{f} \right)_{0,j} \left\{ \left[ (y_{\xi})_{N_{\xi} - \frac{1}{2}, j+\frac{1}{2}} + (y_{\eta})_{N_{\xi} - \frac{1}{2}, j+\frac{1}{2}} \right] \left[ (\delta_j y)_{0,j} + (\delta_j y)_{0,j} \right] \\
+ \left[ (x_{\xi})_{N_{\xi} - \frac{1}{2}, j+\frac{1}{2}} + (x_{\eta})_{N_{\xi} - \frac{1}{2}, j+\frac{1}{2}} \right] \left[ (\delta_i x)_{0,j} + (\delta_i x)_{0,j} \right] \right\} \quad (4.158) \]

\[ a_{0,j+1}^{N_{\xi}-1,j} = \frac{1}{8} \left( \frac{\kappa}{f} \right)_{0,j+1} \left\{ \left[ (y_{\xi})_{N_{\xi} - \frac{1}{2}, j+\frac{1}{2}} - (y_{\eta})_{N_{\xi} - \frac{1}{2}, j+\frac{1}{2}} \right] \left[ (\delta_j y)_{0,j+1} - (\delta_j y)_{0,j+1} \right] \\
+ \left[ (x_{\xi})_{N_{\xi} - \frac{1}{2}, j+\frac{1}{2}} - (x_{\eta})_{N_{\xi} - \frac{1}{2}, j+\frac{1}{2}} \right] \left[ (\delta_i x)_{0,j+1} + (\delta_i x)_{0,j+1} \right] \right\} \quad (4.159) \]

At the points \( j = 0, T_{i+1/2,-1/2} = T_{i+1/2,1/2} \), and so the diagonal elements of each block must be substituted:

\[ a_{i,0}^{i,0} \rightarrow a_{i,0}^{i,0} + a_{i,-1}^{i,0} \quad (4.160) \]

\[ a_{i+1,0}^{i,0} \rightarrow a_{i+1,0}^{i,0} + a_{i+1,-1}^{i,0} \quad (4.161) \]

\[ a_{i-1,0}^{i,0} \rightarrow a_{i-1,0}^{i,0} + a_{i-1,-1}^{i,0} \quad (4.162) \]
and $a_{i-1,N-1}^{i,0}$, $a_{i+1,N-1}^{i,0}$, $a_{i-1,N}^{i,0}$ become undefined.

At $j = N - 1$, $T_{i+1/2,N+1/2} = T_{i+1/2,N-1/2}$,

$$a_{i,N-1}^{i,N-1} = a_{i,N-1}^{i,N-1} + a_{i,N-1}^{i,N-1}$$  (4.163)

$$a_{i+1,N-1}^{i,N-1} = a_{i+1,N-1}^{i,N-1} + a_{i+1,N-1}^{i,N-1}$$  (4.164)

$$a_{i-1,N-1}^{i,N-1} = a_{i-1,N-1}^{i,N-1} + a_{i-1,N-1}^{i,N-1}$$  (4.165)

and $a_{i,N}^{i,N}$, $a_{i+1,N}^{i,N}$, $a_{i-1,N}^{i,N}$ become undefined.

### 4.3.2 1D Preconditioning.

The preconditioner $P$ in 4.58 is built in the way similar to that described in previous section for the matrix $A$, but uses the one-dimensional operator 4.57 instead of 4.129. Consider the elements $p_{ij}^k$ of $P$, where the notation is similar to that of the $a_{ij}^k$. The matrix $P$ is block-diagonal, and so $i = k$ in every case. The elements are:

$$p_{ij}^k = \frac{c_v}{\Delta t} (\rho_0 J_0)_{i+\frac{1}{2},j+\frac{1}{2}} \left[ 1 + (\Gamma - 1) \frac{J_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1} - J_{i+\frac{1}{2},j+\frac{1}{2}}^n}{J_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1} + J_{i+\frac{1}{2},j+\frac{1}{2}}^n} \right]$$

$$+ (x_\xi)_{i+\frac{1}{2},j+\frac{1}{2}} \left[ \frac{\kappa_{i+\frac{1}{2},j+\frac{3}{2}}}{J_{i+\frac{1}{2},j+\frac{3}{2}}} + \frac{\kappa_{i+\frac{1}{2},j+\frac{1}{2}}}{J_{i+\frac{1}{2},j+\frac{1}{2}}} \right] (x_{i+1,j+1} - x_{i,j+1})$$

$$+ \frac{\kappa_{i+\frac{1}{2},j+\frac{3}{2}}}{J_{i+\frac{1}{2},j+\frac{3}{2}}} + \frac{\kappa_{i+\frac{1}{2},j+\frac{1}{2}}}{J_{i+\frac{1}{2},j+\frac{1}{2}}} (x_{i+1,j} - x_{i,j})$$  (4.166)

$$p_{i,j+1}^k = - (x_\xi)_{i+\frac{1}{2},j+\frac{1}{2}} \left[ \frac{\kappa_{i+\frac{1}{2},j+\frac{3}{2}}}{J_{i+\frac{1}{2},j+\frac{3}{2}}} + \frac{\kappa_{i+\frac{1}{2},j+\frac{1}{2}}}{J_{i+\frac{1}{2},j+\frac{1}{2}}} \right] (x_{i+1,j+1} - x_{i,j+1})$$  (4.167)

$$p_{i,j-1}^k = - (x_\xi)_{i+\frac{1}{2},j+\frac{1}{2}} \left[ \frac{\kappa_{i+\frac{1}{2},j+\frac{3}{2}}}{J_{i+\frac{1}{2},j+\frac{3}{2}}} + \frac{\kappa_{i+\frac{1}{2},j+\frac{1}{2}}}{J_{i+\frac{1}{2},j+\frac{1}{2}}} \right] (x_{i+1,j} - x_{i,j})$$  (4.168)
Chapter 5

Conclusions.

This thesis discussed two different aspects of the hydrodynamic stability of inertial confinement fusion implosions: the feedout and the deceleration phase Rayleigh-Taylor instability.

Feedout

The feedout is the process of seeding the acceleration phase Rayleigh-Taylor instability by inner surface nonuniformities. During the initial phase of the implosion, the shell inner surface is stable but the nonuniformities initially present on the inner surface migrate toward the outer surface. This surface is unstable to the R-T instability during the acceleration phase and the perturbations traveling from the inner to the outer surface grow exponentially once they have reached the unstable outer surface. We have calculated both analytically and numerically, the amplitude of the outer surface distortion induced by the initial rear surface nonuniformities. After the onset of the acceleration phase R-T instability, the outer surface distortion $\Delta_{out}$ grows according to the following simple formula

$$\Delta_{outer} = F_c \Delta_0 e^{\int_k^t \gamma(\tau) d\tau}$$  \hspace{1cm} (5.1)
where $\gamma$ is the R-T growth rate, $k$ is the perturbation wavenumber, $\Delta_0$ is the initial rear surface perturbation amplitude, and $F_c$ is the so-called transfer function. For long wavelength modes satisfying $kd_{ps} < 1$ (here $d_{ps}$ is the post-shock thickness), the transfer function $F_c$ has been calculated analytically by solving the ideal gas-dynamic equations neglecting thermal transport. For $kd_{ps} < 1$ the transfer function depends only on the dimensionless wavenumber $kd_{ps}$ and it can be written in the following simple form

$$F_c(kd_{ps}) = \frac{1}{16} \left( \frac{1}{kd_{ps}} + \frac{2.5}{\sqrt{kd_{ps}}} \right) \quad (5.2)$$

Equation 5.2 shows that feedout becomes less effective at shorter wavelength as the transfer function decays as $1/k$. For NIF-like capsule, the modes with substantial feedout have Legendre mode numbers below 30 and the transfer function for modes with $l > 30$ is below 10% ($F_{NIF}(l > 30) < 0.1$). The validity of the long-wavelength theory is confirmed by a series of two-dimensional Lagrangian numerical simulations of the ideal gas dynamic equations. The results of the simulations are in good agreement with the analytical theory as long as $kd_{ps} < 1$.

The short-wavelength feedout ($kd_{ps} > 1$) is investigated numerically. The behavior of short-wavelength modes is strongly affected by ablation and finite thermal conductivity, and the Lagrangian code has been modified to include electronic thermal conduction and laser energy deposition. This is treated as a simple energy source term localized at the critical surface where the plasma frequency equals the laser frequency.

The simulations indicate that the seeding of short-wavelength modes is qualitatively different from the one of the long-wavelength modes as short-wavelength feedout induces temporal oscillations on the outer surface before growing exponentially. The transfer function in the case $kd_{ps} > 1$ is reduced with respect to the
predictions of the long-wavelength theory. Fitting of the numerical results indicate that the deviation of the transfer function from its classical form 2.37 in logarithmic scale is quadratic in the dimensionless parameter $kd_{ps}$. The transfer function valid for short as well as long wavelength modes can be written in the following form:

$$\log (F) = \log (F_c) \left\{ 1 + \alpha (kd_{ps})^2 \right\}$$  \hspace{1cm} (5.3)

where $F_c$ is the classical transfer function 5.2 and $\alpha$ is a parameter depending on the remaining dimensionless parameters. Numerical simulations have shown that the parameter $\alpha$ depends on the dimensionless parameter $L_m/d_{ps}$, where $L_m$ is the density gradient scale length and $d_{ps}$ is the compressed shell thickness. The final form of the transfer function valid for both long and short-wavelength perturbations is

$$\log (F) = \log (F_c) \left\{ 1 + 0.005 (kd_{ps})^2 + 0.038 (kL_m) (kd_{ps}) \right\}$$  \hspace{1cm} (5.4)

Equations 5.2 and 5.4 can be used to determine the R-T induced distortion of the shell outer surface seeded by a rear surface perturbation with initial amplitude $\Delta_0$ and wavenumber $k$.

**Deceleration Phase**

The second part of the thesis deals with the deceleration phase of the ICF implosions. It is shown that the shell is decelerated first by a series of shocks reflecting off the center of the capsule and the inner surface of the shell. After the impulsive deceleration, and the shell is continuously decelerated by the large pressure building up within the hot spot. This latter stage is referred to as the "deceleration phase".

We have investigated both analytically and numerically the one dimensional hot spot and shell dynamics as well as the liner stability of the shell inner surface.
It is found that heat flux leaving the hot spot through thermal conduction is deposited on the shell inner surface. The shell material on its inner surface absorbs the heat, increase its internal energy and expands inward into the hot spot while the heat front travels through the shell with the ablation velocity. The heat lost from the hot spot goes back in the form of internal energy and $pdV$ work of the material ablated off the inner shell surface. Though the hot-spot temperature is reduced by the heat conduction losses, the hot-spot density increases due to the ablated material in such a way that the hot-spot pressure is approximately independent of heat conduction. For locally deposited alpha particles and for temperatures exceeding $\sim 4keV$, heat conduction losses do not affect the onset of the ignition process which occurs after stagnation and only if the alpha heating time scale is shorter than the decompression time of the hot-spot due to the outward motion of the shell. For direct-drive NIF-like capsules, the ablation velocity off the shell inner surface is of the order of tens $\mu m/ns$, the deceleration of the order of thousands $\mu m/ns^2$ and the density-gradient scale-length of the order a few $\mu m$. Using the well established theory of the ablative Rayleigh-Taylor instability, it is shown that the spectrum of the deceleration phase instability exhibits a cutoff for mode numbers of about $l \approx 90$ and all modes with $l > 90$ are suppressed.
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